International Journal of Research in Advent Technology, Vol.6, No.8, August 2018
E-ISSN: 2321-9637
Available online at www.ijrat.org

Application of Memory effects In an Inventory Model with Linear Demand and No shortage

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Abstract-In this paper, our purpose is to bring out the traditional thoughts of the classical inventory model and to overcome some difficulty of the traditional classical inventory model. Classical inventory model is not able to include memory or past experience effects. It is known that a physical meaning of the fractional order is an index of memory. This paper wants to develop three type generalization as fractional order inventory model using fractional calculus according as (i) only the rate of change of inventory level fractional order α ,(ii) demand rate as a fractional polynomial of degree α and α is the rate of change of the inventory level.(iii)demand rate as a fractional polynomial of degree m may be different from the order α .The fractional models are governed by the Caputo fractional order derivative.

Keywords: Euler gamma function; Memory dependent derivative; Fractional order inventory model.

1. INTRODUCTION

In 1695 Gottfried Leibnitz gave a letter to Guillaume L'Hopital with a question was it possible if the order of derivative is some irrational, fractional or complex number? "Dream commands life" and this idea gave a direction to open a new branch of calculus which is named as fractional calculus. The earliest systematic studies were attributed by Leibnitz, Riemann-Liouville for long time. Due to lack of physical interpretation, use of fractional calculus was not realm in applied field. But in recent trends, it is going rapid developments in the applied science because a physical intuition of fractional calculus has been found which signifies index of memory. In ordinary calculus has so many advantages but it is not able to include memory of the system but it is appropriate for fractional calculus. Due to the above advantage of fractional calculus, it is rapidly used to the memory affected system which are affected by memory or past experience otherwise it is not logical. Memory means it depends not only present state of the system but also past state of the system. Some field like biological system[**1**],physics[**2**],financial process[**3**,**4,5**] has great importance of memory effect because there are some endogenous and exogenous variables of the inventory system, are very much depend on the memory or past experience of the system. Inventory system is one of the most wrathful example as memory affected system.

Inventory means stock of goods or resources. Inventory model is formulated for the business purpose to determine the optimum ordering interval with minimized total average cost and optimum level of inventories. Harris was the first person who attributed to develop the EOQ model and Wilson also gave the attention to make up analytical result of the EOQ model. There are listed so many researchers like Dave and patel [6], McDonald[7], Silver and Meal[8],Donaldson[9], Agarwal [10] gave their contribution to give more and more realistic idea about inventory system depending on demand rate, shortage etc.

Here, inventory system is considered as memory affected system. Why?

In inventory system, optimal ordering interval and minimized total average cost depend on some endogenous or exogenous variable of the inventory system. It depends on the environment of the shop or company i.e. position of the shop or company, political or social situation. Moreover, decrease or increase of profit depends on the dealing of the staff of the company or shopkeeper because bad behavior is not suitable to deal with customer properly. On the other hand, if customers gain some poor experience from any company or shop, further they will not agree to purchase products from those companies or shop inspite of its Product's popularity will decrease. Another type of memory is regarded corresponding carrying or holding cost which refers to total cost of holding or carrying the total inventory. There is included transportation system. The dealings of the transportation driver have effect on the business. If it provides poor service then further company or shopkeeper do not want to take this poor service in future. Above all reasons imply that Inventory system is memory affected system.

Here, we have suggested fractional order derivative to take into account memory effect. But why?

It is known that the time rate of change of integer orders are determined by the property of differentiable functions of time only in infinitely small neighborhood of the considered point of time. Hence, there is assumed an instant changes of the marginal output, when the input level changes. Therefore, dynamic memory effect is not present in classical calculus and it is not able to discuss all state of the system i.e (present system depend on the past).But in fractional derivative the rate of change is affected by all points of the considered interval, so it is a memory dependent derivative [1] and fractional order is physically treated as an index of memory. It can remove amnesia from the system as fractional differentiation involves integration over time from past up to the present point of interest or present to future point of interest

In this paper, three type of generalization with fractional calculus has been proposed from the linear type demand rate, no shortage type inventory model. Three fractional order inventory models have been considered as (i) only the rate of change of inventory level fractional order α ,(ii) demand rate as a fractional polynomial of degree α where α is the rate of change of the inventory level.(iii)demand rate as a fractional polynomial of degree m may be different from the order α .Here, left-caputo fractional order derivative has been refered.To solve the fractional differential equation, fractional laplace transform method[14,22] has been used and to evaluate carrying cost , fractional order integration has been applied. Fractional derivative and integration play major role to derive the all fractional model. But in the classical model, ordinary integer order derivative and integration have been applied. It is analogous to the physical meaning of speed. It is also known to us that derivative of integer order is determined by the property of differentiable functions of time only in infinitely small neighborhood of the measured point of time.

Our analysis establishes that Classical inventory system is a particular case of the fractional order inventory system and the business is very much affected for dealings and environment and political and social situation and it is also observed that the system is not so affected by the service of the transportation driver. In the system, once, critical memory effect has been found where business policy falls down.

Rest part of the paper is arranged by the following ways in the section-2,review of fractional calculus, classical inventory model has been given in section3, in the section-3.4, fractional order inventory model has been served, Numerical example has been furnished in the section-4, Some conclusions are cited in the section-5.

2. REVIEW OF FRACTIONAL CALCULUS

2.1 *Euler Gamma Function*

Euler's gamma function is one of the best tools in fractional calculus which was proposed by the Swiss mathematicians Leonhard Euler (1707-1783).The gamma function $\Gamma(x)$is continuous extension from the factorial notation. The gamma function is denoted and defined by the formulae

$$\Gamma(x)=\int_0^\infty t^{(x-1)}e^{-t}dt \qquad x>0$$

$\Gamma(x)$is extended for all real and complex numbers and the gamma function satisfies some basic properties $\Gamma(x+1)=x\Gamma(x)\Rightarrow\Gamma(x)=\frac{\Gamma(x+1)}{x}$, $\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2},\Gamma\left(-\frac{7}{6}\right)=-\frac{6}{7}\Gamma\left(\frac{1}{6}\right)$

Numerically $x!$ can be evaluated for all positive integer values numerically but $\Gamma(x+1)$ can be evaluated for real values.

2.2 *Riemann-Liouville fractional derivative(R-L)*

Left Riemann-Liouville fractional derivative of order α is denoted and defined as follows

$$ {}_aD_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m\int_a^x(x-\tau)^{(m-\alpha-1)}f(\tau)d\tau \quad (1a) $$

where $x>0$

Right Riemann-Liouville fractional derivative of order α is defined as follows

$$ {}_xD_b^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(m-\alpha)}\left(-\frac{d}{dx}\right)^m\int_x^b(x-\tau)^{(m-\alpha-1)}f(\tau)d\tau \quad (1b) $$

where $x>0$

Riemman-Liouville fractional derivative of any constant function is not equal to zero which creates a distance between ordinary calculus and fractional calculus. This definition creates a difficulty that action of derivative of constant term is not zero.

2.3 *Caputo fractional order derivative*

Left Caputo fractional derivative [11] for the function $f(x)$ which has continuous, bounded derivatives in $[a,b]$ is denoted and defined as follows

$$ {}_a^CD_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(m-\alpha)}\int_a^x(x-\tau)^{(m-\alpha-1)}f^m(\tau)d\tau \quad (2) $$

where $0\le m-1<\alpha<m$

Right Caputo fractional derivative for the function $f(x)$ which has continuous and bounded derivatives in $[a,b]$, is defined as follows

$$ {}_x^CD_b^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(m-\alpha)}\int_x^b(\tau-x)^{(m-\alpha-1)}f^m(\tau)d\tau $$

where $0\le m-1<\alpha<m$

$$ {}_a^CD_x^{\alpha}(A)=o,\text{ where } A=\text{constant}. $$

2.4*Fractional Laplace transforms Method*

The Laplace transform of the function $f(t)$ is defined as

$$ F(s)=L(f(t))=\int_0^{\infty}e^{-st}f(t)dt \quad (3a) $$

where s>0 and s is called the transform parameter. The Laplace transformation of n^{th} order derivative is defined as

$$ L\left(f^n(t)\right)=s^nF(s)-\sum_{k=0}^{n-1}s^{n-k-1}f^k(0) \quad (3b) $$

where $f^n(t)$ denotes n^{th} derivative of the function f with respect to t and for non – integer m it is defined in generalized form[**13**] as,

$$ L\left(f^m(t)\right)=s^mF(s)-\sum_{k=0}^{n-1}s^kf^{m-k-1}(0) \quad (3c) $$

Where m is the largest integer such that $(n-1)<m\le n.$

2.5 *Memory dependent derivative*

Derivative of any function using the kernel can be written in the following form [**1**]

$$ D\left(f(x)\right)=\int_a^xK(x-s)f'(s)ds \quad (4a) $$

For integer order derivative the kernel is considered as $K(x-s)=\delta(x-s)$ and it gives the memory less derivative. To derive the concept of memory effect using definition (4a) we consider $K(x-s)=\frac{(x-s)^{m-\alpha}}{\Gamma(m-\alpha)}$ and expressed in the following form

$$ D_a^{\alpha}\left(f(x)\right)=\int_a^xK(x-s)f^m(s)ds \quad (4b) $$

Where f^m denotes the common *m-th* order derivative, which has specific physical meaning. The integer order derivative is a local property but the $\alpha-th$ order fractional derivative is not a local property. The total effects of the commonly used $\alpha-th$ derivative on the interval *[a, x]* describes the variation of a system in which the instantaneous change rate depends on the past state, is called the ‘‘memory effect’’. Here, the rate of memory kernel decays depending on α. Hence, the strength of the memory is controlled by α. When, $\alpha\to1$, it becomes week in the sense of memory and when $\alpha=1.0$, the system becomes memory less. The low value of α indicates long memory of the system.

3. MODEL FORMULATIONS

In this section, the classical inventory model has been introduced analytically. All mathematical models in this paper are developed on the basis of the following notations and assumptions.

3.1 *Notations*

$(i)R$: Demand rate	$(ii)Q$: Total order quantity
$(iii)U$: Per unit cost	$(iv)C_1$: Inventory holding cost per unit
$(v)C_3$: Ordering cost or setup cost	$(vi)I(t)$: Stock level or inventory level
$(vii)T$: Ordering interval.	$(viii)HOC$: Inventory holding cost per cycle for the classical inventory model.
$(ix)T^*$: Optimal ordering interval	$(x)TOC$: Total average cost during the total time interval
$(xi)TOC*$: Minimized total average cost during the total time interval $[0,T]$ for classical inventory model.	$(xii)TOC^*_{\alpha,\beta}$: Minimized total average cost during the total time interval $[0,T]$ for fractional order model as defined in the section-3.4.1.
$(xiii)T^*_{\alpha,\beta}$: Optimal ordering interval for fractional order inventory model as defined in the section-3.4.1.	$(xiv)HOC_{\alpha,\beta}$: Inventory holding cost per cycle for fractional order inventory model as defined in the section 3.4.1.
$(xv)T^*_{\alpha,m,\beta}$: Optimal ordering interval for fractional order inventory model as defined in section-3.4.3.	$(xvi)TOC^*_{\alpha,m,\beta}$: Minimized total average cost during the total time interval $[0,T]$ as defined in section-3.4.3.
$(xvii)HOC_{\alpha,m,\beta}$: Inventory holding cost per cycle for fractional order inventory model as defined in section-3.4.3.	$(xviii)B$: Beta function.
$(xix)T^*_{\alpha,\beta}$: Optimal ordering interval for fractional order inventory model as defined in the section-3.4.2.	$(xx)HOC_{\alpha,\beta}$: Inventory holding cost per cycle for fractional order inventory model as defined in the section-3.4.2.
$(xxi)TOC^*_{\alpha,\beta}$: Minimized total average cost during the total time interval $[0,T]$ as defined in section-3.4.2.	$(xxii)\Gamma$: gamma function.

Table-1: Used different symbols and items for developing the models.

3.2*Assumptions*

(i) Lead time is zero.
(ii)Time horizon is infinite.
(iii)There is no shortage.
(iv)There is no deterioration.

3.3 *Classical inventory model*

During the period $[0,T]$, the inventory level depletes due to the demand with demand rate $(a+bt), a,b>0$ where shortage is not allowed. Hence, the ordinary differential equation governing the inventory level at any time *t* during the period *[0,T]* is given by

$$\frac{d\left(I(t)\right)}{dt} = -\left(a+bt\right) \quad for\, 0 \le t \le T \tag{5}$$

with boundary conditions are $I(T)=0$ and $I(0)=Q$. Inventory level is obtained by solving (5) with the boundary condition $I(T)=0$ in the following form

$$I(t) = a(T-t) + \frac{b}{2}\left(T^2 - t^2\right) \tag{6}$$

Using initial condition $I(0)=Q$, the optimal order quantity can be obtained as,

$$I(t) = \left(aT + \frac{b}{2}T^2\right) \tag{7}$$

Corresponding total inventory holding cost over the time interval *[0,T]* is

$$\begin{aligned} HOC\left(T\right) &= C_1 \int_0^T \left(a\left(T-t\right) + \frac{b}{2}\left(T^2 - t^2\right)\right) dt \\ &= C_1 \left(\frac{aT^2}{2} + \frac{bT^3}{3} \right) \end{aligned} \tag{8}$$

Hence, the total cost per unit time is as

$TOC(T)$= ((Purchasing cost (PC)) +(holding cost($HOC(T)$))+(set up cost (C_3)))

$$TOC(T) = \left(U\left(aT+\frac{b}{2}T^2\right)+C_1\left(\frac{aT^2}{2}+\frac{bT^3}{3}\right)+C_3\right) \quad (9)$$

Therefore, the total average cost per unit time per cycle is

$$TOC_{av}(T) = \frac{\left(U\left(aT+\frac{b}{2}T^2\right)+C_1\left(\frac{aT^2}{2}+\frac{bT^3}{3}\right)+C_3\right)}{T} \quad (10)$$

Thus, the objective of the classical EOQ model can be represented in the form as,

$$\begin{cases} MinTOC(T) = \dfrac{(UQ+HOC(T)+C_3)}{T} \\ Subject\ to\ T \geq 0 \end{cases} \quad (11)$$

We now want to find its optimal ordering interval T^* using the necessary condition (i) $\frac{d(TOC)}{dT}=0$ provided (ii) $\left.\frac{d^2(TOC)}{dT^2}\right|_{T=T^*} > 0$.

The necessary condition (i) gives

$$\left(\frac{bU}{2}+\frac{aC_1}{2}\right)-\frac{C_3}{T^2}+\frac{bC_1T^2}{3}=0 \quad (12)$$

The optimal ordering interval T^* has been found from the non-linear polynomial equation. Thus TOC^* is the minimum value of TOC (T^*) which is obtained at this optimal ordering interval T^*.

3.4Fractional Generalization of the Classical EOQ Models

Now, we are going to generalize the above classical inventory model considering fractional order rate of change of the inventory level with different type demand rate **(i)** quadratic type polynomial **(ii)** fractional order polynomial whose order is same as fraction order rate of change of inventory level, **(iii)** fractional order polynomial whose order may not be same as fractional order rate of change of inventory level.

Model-I formulation with memory kernel

In order to study the study of memory effect on the inventory model, we consider the fractional generalization of the classical inventory model or memory less inventory model with linear type demand rate $(a+bt)$and no shortage. All other assumptions are same as for the classical inventory model. Due to observe the influence of memory effects, first the differential equation (5) can be written using the kernel function as follows [**1**].

$$\frac{dI(t)}{dt} = -\int_0^t k(t-t')(a+bt')dt' \quad (13)$$

in which $k(t-t')$ is the kernel function. For Markov process it is equal to the delta function $\delta(t-t')$ and it will generate the equation (5). Indeed, any arbitrary function can be succeed by a sum of delta functions, thereby leading to a given type of time correlations. An appropriate choice, in order to include in long-term memory effects, can be a power-law function which displays a slow decay such that the state of the system at quite early times also contributes to the evolution of the system[**1**].

This type of kernel guarantees the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order model we consider $k(t-t') = \frac{(t-t')^{(\alpha-2)}}{\Gamma(1-\alpha)}$, where $0<\alpha\leq 1$ and $\Gamma(\alpha)$ denotes the gamma function. Using the definition of fractional derivative [**13,14**] we can re-write the Equation (13) to the form of fractional differential equations with the Caputo-type derivative in the following form

$$\frac{dI(t)}{dt} = -{}_0D_t^{-(\alpha-1)}(a+bt) \quad (14)$$

Now, applying fractional Caputo derivative of order $(\alpha-1)$ on both sides of the Eq. (14), and using the fact the Caputo fractional derivative and fractional integral are inverse operators, the following fractional differential equations can be obtained for the model

$${}_0^CD_t^{\alpha} = -(a+bt)$$

$${}_{0}^{C}D_{t}^{\alpha}\left(I(t)\right)=\frac{d^{\alpha}\left(I(t)\right)}{dt^{\alpha}}=-\left(a+bt\right) \quad (15)$$

$$0<\alpha\leq 1,\ 0\leq t\leq T$$

with boundary conditions $I(T)=0$ and $I(0)=Q$.

The strength of memory is controlled by α. When, $\alpha\to 1$, memory of the system becomes weak.

Model-II

In the second fractional model, both rate of change of the inventory level and the demand rate has been generalized. Fractional index in both the cases remains same.

We consider a fractional order inventory model where the rate of change of the inventory level is the fractional type and the demand rate is the fractional type with the same index. The model can be posed as,

$${}_{0}^{C}D_{t}^{\alpha}\left(I(t)\right)=\frac{d^{\alpha}\left(I(t)\right)}{dt^{\alpha}}=-\left(a+bt^{\alpha}\right) \quad (16)$$

$$0<\alpha\leq 1,\ 0\leq t\leq T$$

With boundary conditions are $I(T)=0$ and $I(0)=Q$.

Model-III

In the third fractional model, the rate of change of the inventory level and the demand rate are both generalized but their fractional indexes are considered different. The inventory problem will then be,

$${}_{0}^{C}D_{t}^{\alpha}\left(I(t)\right)=\frac{d^{\alpha}\left(I(t)\right)}{dt^{\alpha}}=-\left(a+bt^{m}\right) \quad (17)$$

$$where,\ 0<\alpha,m\leq 1,\ 0\leq t\leq T$$

with boundary conditions $I(T)=0$ and $I(0)=Q$.

In all the above three generalized cases of fractional order differential equations, boundary conditions are used same as of the classical differential equation. This is the right platform for the use of left-Caputo fractional derivative. Hence, we have been used it and get memory effect of the system.

3.4.1 Analytic solution of model-I

Here, we consider the fractional order inventory model-**I**, which can be solved by using Laplace transform method with the initial condition, given in the problem. In operator form the fractional differential equation in (**15**) can be represented as

$${}_{0}^{C}D_{t}^{\alpha}\left(I(t)\right)=-\left(a+bt\right) \quad (18)$$

$$D^{\alpha}\equiv\frac{d^{\alpha}}{dt^{\alpha}},\ where,\ 0<\alpha\leq 1,\ 0\leq t\leq T$$

Using Laplace transform and the corresponding inversion formula on the equation (18) we get the inventory level for this fractional order inventory model at time *t* which can be written as

$$I(t)=\left(Q-\frac{at^{\alpha}}{\Gamma(1+\alpha)}-\frac{bt^{\alpha+1}}{\Gamma(2+\alpha)}\right) \quad (19)$$

Using the boundary condition $I(T)=0$ on the equation (19) we get the total order quantity as

$$Q=\frac{aT^{\alpha}}{\Gamma(1+\alpha)}+\frac{bT^{\alpha+1}}{\Gamma(2+\alpha)} \quad (20)$$

and corresponding the inventory level at time *t* being,

$$I(t)=\left(\frac{a}{\Gamma(1+\alpha)}\left(T^{\alpha}-t^{\alpha}\right)+\frac{b}{\Gamma(2+\alpha)}\left(T^{\alpha+1}-t^{\alpha+1}\right)\right) \quad (21)$$

The β^{th} $\left(0<\beta\leq 1\right)$ order total inventory holding cost is denoted as $HOC_{\alpha,\beta}(T)$ and defined as

$$\begin{aligned}HOC_{\alpha,\beta}(T)&=C_1\left({}_{0}D_{T}^{-\beta}\left(I(t)\right)\right)\\&=\frac{C_1}{\Gamma(\beta)}\int_0^T (T-t)^{(\beta-1)}\left(\frac{a}{\Gamma(1+\alpha)}\left(T^{\alpha}-t^{\alpha}\right)+\frac{b}{\Gamma(2+\alpha)}\left(T^{\alpha+1}-t^{\alpha+1}\right)\right)dt\\&=\frac{C_1aT^{(\alpha+\beta)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right)\\&+\frac{C_1bT^{(\alpha+\beta+1)}}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right) \quad (22)\end{aligned}$$

(here, β is considered another memory parameter corresponding carrying cost. It is known to us that carrying cost is referred to the total cost for carrying or holding of the total inventory. Hence, it is transportation related cost. In the transportation system, transportation driver may be good or bad. The effect of bad service always has a bad impact on the business due to the above reason a memory effect will be found on the business)

Therefore, the total average cost per unit time per cycle is

$$TOC_{\alpha,\beta}(T) = \frac{\left(UQ + HOC_{\alpha,\beta}(T) + C_3\right)}{T}$$

$$= \left(\begin{array}{l} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bUT^{\alpha}}{\Gamma(2+\alpha)} \\ + \frac{C_1 a T^{(\alpha+\beta-1)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right) \\ + \frac{C_1 b T^{(\alpha+\beta)}}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right) + C_3 T^{-1} \end{array} \right) \quad (23)$$

Here, four cases have been studied for the characterization of this fractional order inventory model (i) $0<\alpha\le 1.0, 0<\beta\le 1.0$,

(ii) $\beta = 1.0$ and $0<\alpha\le 1.0$,

(iii) $\alpha = 1.0$ and $0<\beta\le 1.0$,

(iv) $\alpha = 1.0, \beta = 1.0$.

(i)Case-1: $0<\alpha\le 1.0,\ 0<\beta\le 1.0.$

In this case, the total average cost becomes

$$TOC^{av}_{\alpha,\beta}(T) = \left(\begin{array}{l} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bUT^{\alpha}}{\Gamma(2+\alpha)} \\ + \frac{C_1 a T^{(\alpha+\beta-1)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right) \\ + \frac{C_1 b T^{(\alpha+\beta)}}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right) \\ + C_3 T^{-1} \end{array} \right) \quad (24)$$

To evaluate the minimum value of the total average cost $TOC^{av}_{\alpha,\beta}(T)$, we propose the corresponding non-linear programming problem in the following form and solve primal geometric programming method, are discussed bellow

$$MinTOC^{av}_{\alpha,\beta}(T) = \begin{cases} AT^{(\alpha-1)} + B_1 T^{\alpha} + CT^{\alpha+\beta-1} + DT^{(\alpha+\beta)} + ET^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (25)$$

$$\left(\begin{array}{l} \text{Where}, A = \frac{aU}{\Gamma(1+\alpha)},\ B_1 = \frac{bU}{\Gamma(2+\alpha)}, \\ C = \frac{C_1 a}{\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\beta)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right), \\ D = \frac{C_1 b}{\Gamma(2+\alpha)}\left(\frac{1}{\Gamma(1+\beta)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right), \\ E = C_3 \end{array} \right)$$

(A)Primal Geometric programming method

The above inventory model(25) is solved by primal geometric programming method [22,26].

The dual form of the aboveprimal(25) model is as,

$$Maxd(w) = \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5} \quad (26)$$

Orthogonal condition is as

$$(\alpha-1)w_1 + (\alpha)w_2 + (\alpha+\beta-1)w_3 + (\alpha+\beta)w_4 - w_5 = 0 \quad (27)$$

Normalized condition is as

$$w_1 + w_2 + w_3 + w_4 + w_5 = 1 \quad (28)$$

Primal –dual relations are

$$\left(\begin{array}{l} AT^{(\alpha-1)} = w_1 d(w), B_1 T^{\alpha} = w_2 d(w), \\ CT^{\alpha+\beta-1} = w_3 d(w), \\ DT^{(\alpha+\beta)} = w_4 d(w), \\ ET^{-1} = w_5 d(w) \end{array} \right)$$

From the above primal dual relations we get,

$$\begin{pmatrix} \left(\dfrac{Aw_2}{B_1 w_1}\right)^{(\beta-1)} = \left(\dfrac{B_1 w_3}{C w_2}\right), \\ \left(\dfrac{Aw_2}{B_1 w_1}\right) = \left(\dfrac{C w_4}{D w_3}\right), \\ \left(\dfrac{Aw_2}{B_1 w_1}\right)^{(\alpha+\beta-1)} = \left(\dfrac{E w_4}{D w_5}\right) \end{pmatrix} \quad (29)$$

Along with,

$$T = \left(\frac{Aw_2}{B_1 w_1}\right) \quad (30)$$

The Non-linear equations (27,28,29) for w_1, w_2, w_3, w_4, w_5 can be solved to obtain $w_1^*, w_2^*, w_3^*, w_4^*, w_5^*$.Optimal ordering interval and minimized total average cost can be solved from (30) and(25) analytically.

(ii)Case-2: $\beta = 1.0$ and $0 < \alpha \leq 1.0.$

Therefore, total average cost in this case is as follows

$$TOC_{\alpha,1}^{av}(T) = \begin{pmatrix} \dfrac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \dfrac{bUT^{\alpha}}{\Gamma(2+\alpha)} \\ + \dfrac{C_1 a T^{(\alpha)}}{\Gamma(\alpha+1)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+1,1)}{1}\right) \\ + \dfrac{C_1 b T^{(\alpha+1)}}{\Gamma(\alpha+2)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+2,1)}{\Gamma(1)}\right) \\ + C_3 T^{-1} \end{pmatrix} \quad (31)$$

In this case, the generalized inventory model (**25**) will be in the following form,

$$\text{Min} TOC_{\alpha,1}^{av}(T) = \begin{cases} AT^{(\alpha-1)} + B_1 T^{\alpha} + CT^{\alpha} + DT^{(\alpha+1)} + ET^{-1} \\ T \geq 0 \end{cases} \quad (32)$$

$$\begin{pmatrix} \text{Where, } A = \dfrac{aU}{\Gamma(1+\alpha)},\ B_1 = \dfrac{bU}{\Gamma(2+\alpha)}, \\ C = \dfrac{C_1 a}{\Gamma(1+\alpha)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+1,1)}{\Gamma(1)}\right), \\ D = \dfrac{C_1 b}{\Gamma(2+\alpha)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+2,1)}{\Gamma(1)}\right), \\ E = C_3 \end{pmatrix} \quad (33)$$

In the similar manner as in case (i) of model-I, primal geometric programming algorithm can provide the minimized total average cost and optimal ordering interval $TOC_{\alpha,1}^*$, $T_{\alpha,1}^*$.

(iii)Case-3: $\alpha = 1.0$ and $0 < \beta \leq 1.0.$

Total average cost per unit time per cycle is,

$$TOC_{1,\beta}^{av}(T) = \begin{pmatrix} \dfrac{aUT^{0}}{\Gamma(2)} + \dfrac{bUT^{1}}{\Gamma(3)} \\ + \dfrac{C_1 a T^{(\beta)}}{\Gamma(2)}\left(\dfrac{1}{\Gamma(\beta+1)} - \dfrac{B(2,\beta)}{\Gamma(\beta)}\right) \\ + \dfrac{C_1 b T^{(1+\beta)}}{\Gamma(3)}\left(\dfrac{1}{\Gamma(\beta+1)} - \dfrac{B(3,\beta)}{\Gamma(\beta)}\right) \\ + C_3 T^{-1} \end{pmatrix}$$

In this case, the generalized inventory model (25) becomes as,

$$\begin{cases} \text{Min } TOC_{1,\beta}^{av}(T) = AT^{0} + B_1 T + CT^{\beta} + DT^{1+\beta} + ET^{-1} \\ \text{Subject to} \quad T \geq 0 \end{cases} \quad (34)$$

$$\text{where,} \begin{pmatrix} A = \dfrac{aU}{\Gamma(2)}, B_1 = \dfrac{bU}{\Gamma(3)}, \\ C = \dfrac{C_1 a}{\Gamma(2)}\left(\dfrac{1}{\Gamma(\beta+1)} - \dfrac{B(2,\beta)}{\Gamma(\beta)}\right), \\ D = \dfrac{C_1 b}{\Gamma(3)}\left(\dfrac{1}{\Gamma(\beta+1)} - \dfrac{B(3,\beta)}{\Gamma(\beta)}\right), \\ E = C_3 \end{pmatrix}$$

In the similar manner as in case (i) of model-I, primal geometric programming algorithm can provide the minimized total average cost and optimal ordering interval $TOC_{1,\beta}^*$, $T_{1,\beta}^*$. respectively

(iv) Case-4: $\beta = 1.0,\ \alpha = 1.0.$

In this case, the generalized inventory model (25) will be,

$$\begin{cases} Min\,TOC_{1,1}^{av}(T) = AT^0 + B_1T + CT^1 + DT^2 + ET^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (35)$$

$$\left(\begin{array}{l} \text{Where, } A = \frac{aU}{\Gamma(2)}, B_1 = \frac{bU}{\Gamma(3)}, \\ C = \frac{C_1 a}{\Gamma(2)}\left(\frac{1}{\Gamma(2)} - \frac{B(2,1)}{\Gamma(1)}\right), \\ D = \frac{C_1 b}{\Gamma(3)}\left(\frac{1}{\Gamma(2)} - \frac{B(3,1)}{\Gamma(1)}\right), \\ E = C_3 \end{array} \right)$$

The expression (35) coincides to the expression (11) for $\beta = 1.0,\ \alpha = 1.0$.

In the similar way as in case (i) of model-I, primal geometric programming algorithm can provide the minimized total average cost $TOC_{1,1}^{*}(T)$ and optimal ordering interval $T_{1,1}^{*}$.

3.4.2 Analytic solution of model-II

Here, we consider a fractional order inventory model, is described by the equation (16) where the rate of change of the inventory level $I(t)$ is of fractional order α and the demand is a fractional polynomial of order α. This fractional order differential equation has been solved using Laplace transform method. In operator form the equation (16) becomes as,

$$D^\alpha (I(t)) = -(a + bt^\alpha),$$

$$D^\alpha \equiv \frac{d^\alpha}{dt^\alpha}, \quad where,\ 0 < \alpha \le 1,\ 0 \le t \le T \quad (36)$$

Using Laplace transform and the corresponding inversion formula on the equation (36) we get the inventory level for this fractional order inventory model at time *t* which can be written as

$$I(t) = \left(Q - \frac{at^\alpha}{\Gamma(1+\alpha)} - \frac{b\Gamma(1+\alpha)t^{\alpha+1}}{\Gamma(1+2\alpha)} \right) \quad (37)$$

Using the boundary condition $I(T) = 0$ on the equation (37), the total order quantity is obtained as,

$$Q = \frac{aT^\alpha}{\Gamma(1+\alpha)} + \frac{b\Gamma(1+\alpha)T^{\alpha+1}}{\Gamma(1+2\alpha)} \quad (38)$$

The inventory level becomes as,

$$I(t) = \left(\frac{a}{\Gamma(1+\alpha)}\left(T^\alpha - t^\alpha\right) + \frac{b\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(T^{\alpha+1} - t^{\alpha+1}\right) \right) \quad (39)$$

The β^{th} $(0 < \beta \le 1)$ order total fractional inventory holding cost is denoted $HOC_{\alpha,\beta}(T)$ and defined as

$$HOC_{\alpha,\beta}(T) = C_1\left({}_0D_T^{-\beta}(I(t)) \right) = \frac{C_1}{\Gamma(\beta)} \int_0^T (T-t)^{(\beta-1)} \left(\begin{array}{l} \frac{a}{\Gamma(1+\alpha)}\left(T^\alpha - t^\alpha\right) \\ + \frac{b\Gamma(1+\alpha)}{\Gamma(2\alpha+1)}\left(T^{\alpha+1} - t^{\alpha+1}\right) \end{array} \right) dt$$

$$= \left(\begin{array}{l} \frac{C_1 a T^{(\alpha+\beta)}}{\Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)} \right) \\ + \frac{C_1 b \Gamma(\alpha+1) T^{(\alpha+\beta+1)}}{\Gamma(2\alpha+1)} \left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)} \right) \end{array} \right) \quad (40)$$

Therefore, the total average cost is denoted and defined as

$$TOC_{\alpha,\beta}(T) = \frac{\left(UQ + HOC_{\alpha,\beta}(T) + C_3\right)}{T}$$

$$= \left(\begin{array}{l} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bU\Gamma(1+\alpha)T^\alpha}{\Gamma(2\alpha+1)} + \frac{C_1 a T^{(\alpha+\beta-1)}}{\Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)} \right) \\ + \frac{C_1 b \Gamma(1+\alpha) T^{(\alpha+\beta)}}{\Gamma(2\alpha+1)} \left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)} \right) + C_3 T^{-1} \end{array} \right) \quad (41)$$

Now, we shall consider four cases to study the behavior of this fractional order model

$(i) 0<\alpha \le 1.0$ and $0<\beta \le 1.0$
$(ii) \beta = 1.0$ and $0<\alpha \le 1.0$
$(iii)\ \alpha = 1.0$ and $0<\beta \le 1.0$
$(iv) \alpha = 1.0$ and $\beta = 1.0$.

(i)Case-1: $0<\beta \le 1.0,\ 0<\alpha \le 1.0.$

To find the minimum value of the total average cost $TOC_{\alpha,\beta}^{av}(T)$, the corresponding non-linear programming problem can be used in the following form(model-1,case-(i)) and then solve it analytically.

$$MinTOC_{\alpha,\beta}^{av}(T) = \begin{cases} AT^{(\alpha-1)} + B_1T^{\alpha} + CT^{\alpha+\beta-1} + DT^{(\alpha+\beta)} + ET^{-1} \\ T \ge 0 \end{cases} \quad (42)$$

$$\left(\begin{array}{l} \text{Where, } A = \dfrac{aU}{\Gamma(1+\alpha)},\ B_1 = \dfrac{bU\Gamma(1+\alpha)}{\Gamma(2\alpha+1)}, \\ C = \dfrac{C_1a}{\Gamma(1+\alpha)}\left(\dfrac{1}{\Gamma(1+\beta)} - \dfrac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right), \\ D = \dfrac{C_1b\Gamma(1+\alpha)}{\Gamma(2\alpha+1)}\left(\dfrac{1}{\Gamma(1+\beta)} - \dfrac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right), \\ E = C_3 \end{array} \right)$$

(ii) Case-2: $\beta = 1.0,\ 0<\alpha \le 1.0$

In this case, total average cost is presented as follows,

$$TOC_{\alpha,1}(T) = \frac{(UQ + HOC_{\alpha,1}(T) + C_3)}{T}$$
$$= \left(\begin{array}{l} \dfrac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \dfrac{bU\Gamma(1+\alpha)T^{\alpha}}{\Gamma(2\alpha+1)} \\ + \dfrac{C_1aT^{(\alpha)}}{\Gamma(\alpha+1)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+1,1)}{\Gamma(1)}\right) \\ + \dfrac{C_1b\Gamma(1+\alpha)T^{(\alpha+1)}}{\Gamma(2\alpha+1)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+2,1)}{\Gamma(1)}\right) \\ + C_3T^{-1} \end{array} \right) \quad (43)$$

Therefore, the fractional order inventory model (42) in this case will be,

$$\text{Min}TOC_{\alpha,1}^{av}(T) = \begin{cases} AT^{(\alpha-1)} + B_1T^{\alpha} + CT^{\alpha+1} + DT^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (44)$$

$$\left(\begin{array}{l} \text{Where, } A = \dfrac{aU}{\Gamma(1+\alpha)},\ B_1 = \left(\begin{array}{l} \dfrac{bU\Gamma(1+\alpha)}{\Gamma(2\alpha+1)} \\ + \dfrac{C_1a}{\Gamma(1+\alpha)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+1,1)}{\Gamma(1)}\right) \end{array} \right), \\ C = \dfrac{C_1b\Gamma(1+\alpha)}{\Gamma(2\alpha+1)}\left(\dfrac{1}{\Gamma(2)} - \dfrac{B(\alpha+2,1)}{\Gamma(1)}\right),\ D = C_3 \end{array} \right)$$

(iii) Case-3: $\alpha = 1.0, 0<\beta \le 1.0.$

In this case, total average cost becomes as follows

$$TOC_{1,\beta}(T) = \frac{(UQ + HOC_{1,\beta}(T) + C_3)}{T}$$
$$= \left(\begin{array}{l} \dfrac{aUT^{0}}{\Gamma(1+\alpha)} + \dfrac{bU\Gamma(2)T^{1}}{\Gamma(3)} + \dfrac{C_1aT^{(\beta)}}{\Gamma(2)}\left(\dfrac{1}{\Gamma(\beta+1)} - \dfrac{B(2,\beta)}{\Gamma(\beta)}\right) \\ + \dfrac{C_1b\Gamma(2)T^{(1+\beta)}}{\Gamma(3)}\left(\dfrac{1}{\Gamma(\beta+1)} - \dfrac{B(3,\beta)}{\Gamma(\beta)}\right) + C_3T^{-1} \end{array} \right) \quad (45)$$

Therefore, the equation (42) reduces as,

$$\text{Min}TOC_{1,\beta}^{av}(T) = \begin{cases} AT^{(0)} + B_1T^{1} + CT^{\beta} + DT^{1+\beta} + ET^{-1} \\ Subject\ to\ T \ge 0 \end{cases} \quad (46)$$

$$\left(\begin{array}{l} \text{Where, } A = \dfrac{aU}{\Gamma(2)},\ B_1 = \dfrac{bU\Gamma(2)}{\Gamma(3)}, \\ C = \dfrac{C_1a}{\Gamma(2)}\left(\dfrac{1}{\Gamma(1+\beta)} - \dfrac{B(2,\beta)}{\Gamma(\beta)}\right), \\ D = \dfrac{C_1b\Gamma(2)}{\Gamma(3)}\left(\dfrac{1}{\Gamma(1+\beta)} - \dfrac{B(3,\beta)}{\Gamma(\beta)}\right) \\ E = C_3 \end{array} \right)$$

In the similar way as in case (i) of model-I, minimum value of the total average cost $TOC_{1,\beta}^{*}$ and optimal ordering interval and $T_{1,\beta}^{*}$ analytically.

(iv) Case-4: $\alpha = 1.0, \beta = 1.0.$

Therefore, the total average cost is as follows

$$TOC_{1,1}(T) = \frac{(UQ + HOC_{1,1}(T) + C_3)}{T} = \begin{pmatrix} \frac{aUT^0}{\Gamma(2)} + \frac{bU\Gamma(2)T^1}{\Gamma(3)} \\ + \frac{C_1 a T^{(1)}}{\Gamma(2)}\left(\frac{1}{\Gamma(2)} - \frac{B(2,1)}{\Gamma(1)}\right) \\ + \frac{C_1 b\Gamma(2)T^{(2)}}{\Gamma(3)}\left(\frac{1}{\Gamma(2)} - \frac{B(3,1)}{\Gamma(1)}\right) \\ + C_3 T^{-1} \end{pmatrix}$$

Therefore, the equation (42) reduces as,

$$\text{Min}TOC_{1,1}^{av}(T) = \begin{cases} AT^{(0)} + B_1 T^1 + CT^2 + DT^{-1} \\ \text{Subject to } T \geq 0 \end{cases} \quad (47)$$

Where, $A = \frac{aU}{\Gamma(2)}$,

$$B_1 = \begin{pmatrix} \frac{bU\Gamma(2)}{\Gamma(3)} \\ + \frac{C_1 a}{\Gamma(2)}\left(\frac{1}{\Gamma(2)} - \frac{B(2,1)}{\Gamma(1)}\right) \end{pmatrix},$$

$$C = \frac{C_1 b\Gamma(2)}{\Gamma(3)}\left(\frac{1}{\Gamma(2)} - \frac{B(3,1)}{\Gamma(1)}\right),$$

$$D = C_3$$

The expression (47) coincides to the expression (11) for $\alpha = 1.0, \beta = 1.0.$

In the similar manner as in case (i) of model-I, primal geometric programming algorithm can provide the minimum value of the total average cost $TOC_{1,1}^*(T)$ and optimal ordering interval $T_{1,1}^*$.

3.4.3Analytic solution of model –III

Here, we consider the fractional order inventory model which is described by the equation (17).The fractional order differential equation (17) can be solved by using Laplace transform method with the initial condition, are given in the problem. In operator form the equation (17) becomes,

$$D^\alpha(I(t)) = -(a + bt^m),$$

$$D^\alpha \equiv \frac{d^\alpha}{dt^\alpha}, 0 < \alpha \leq 1, 0 \leq t \leq T \quad (48)$$

Using Laplace transform and the corresponding inversion formula on the equation (48), the inventory level at time t can be obtained as,

$$I(t) = \left(Q - \frac{at^\alpha}{\Gamma(1+\alpha)} - \frac{b\Gamma(1+m)t^{\alpha+m}}{\Gamma(1+\alpha+m)}\right) \quad (49)$$

where α may be different from m, $0 < \alpha \leq 1.0$.

where α is the memory parameter

After using the boundary condition $I(T) = 0$ in the equation (49), the total order quantity for this type fractional order inventory model can be obtained as,

$$Q = \frac{aT^\alpha}{\Gamma(1+\alpha)} + \frac{b\Gamma(1+m)T^{\alpha+m}}{\Gamma(1+\alpha+m)} \quad (50)$$

Corresponding memory dependent β^{th} order total inventory holding cost is

$$HOC_{\alpha,m,\beta}(T) = C_1\left({}_0D_T^{-\beta}(I(t))\right) = \frac{C_1}{\Gamma(\beta)}\int_0^T (T-t)^{(\beta-1)}\begin{pmatrix} \frac{a}{\Gamma(1+\alpha)}(T^\alpha - t^\alpha) \\ + \frac{b\Gamma(1+m)}{\Gamma(\alpha+m+1)}(T^{\alpha+m} - t^{\alpha+m}) \end{pmatrix}dt$$

$$= \begin{pmatrix} \frac{C_1 a T^{(\alpha+\beta)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right) \\ + \frac{C_1 b\Gamma(m+1)T^{(\alpha+\beta+m)}}{\Gamma(\alpha+m+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+m+1,\beta)}{\Gamma(\beta)}\right) \end{pmatrix} \quad (51)$$

Therefore, the total average cost is as,

$$TOC_{\alpha,m,\beta}(T) = \frac{(UQ + HOC_{\alpha,m,\beta}(T) + C_3)}{T} = \begin{pmatrix} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bU\Gamma(1+m)T^{\alpha+m-1}}{\Gamma(\alpha+m+1)} \\ + \frac{C_1 a T^{(\alpha+\beta-1)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right) \\ + \frac{C_1 b\Gamma(1+m)T^{(\alpha+\beta+m-1)}}{\Gamma(\alpha+m+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+m+1,\beta)}{\Gamma(\beta)}\right) \\ + C_3 T^{-1} \end{pmatrix} \quad (52)$$

Now, we shall consider eightcases to study this type fractional order model as follows

$(i)0 < m \le 1.0\ ,0 < \alpha \le 1.0, 0 < \beta \le 1.0,$

$(ii) m = 1.0\ and\ 0 < \alpha \le 1.0, 0 < \beta \le 1.0,$

$(iii)\ \beta = 1.0\ and\ 0 < m \le 1.0, 0 < \alpha \le 1.0,$

$(\text{iv})\alpha = 1.0\ and\ 0 < m \le 1.0, 0 < \beta \le 1.0$

$(v) m = \beta = 1.0\ and\ 0 < \alpha \le 1.0,$

$(vi) m = \alpha = 1.0\ and\ 0 < \beta \le 1.0,$

$(vii)\alpha = \beta = 1.0\ , 0 < m \le 1.0,$

$(\text{viii}\ \alpha = m = \beta = 1.0\ .$

(i)Case-1: $0 < \alpha \le 1.0\ , 0 < m \le 1.0, 0 < \beta \le 1.0.$

Total average cost is as,

$$TOC_{\alpha,m,\beta}(T) = \frac{\left(UQ + HOC_{\alpha,m,\beta}(T) + C_3\right)}{T} = \left(\begin{array}{l} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bU\Gamma(1+m)T^{\alpha+m-1}}{\Gamma(\alpha+m+1)} + \frac{C_1 aT^{(\alpha+\beta-1)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right) \\ + \frac{C_1 b\Gamma(1+m)T^{(\alpha+\beta+m-1)}}{\Gamma(\alpha+m+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+m+1,\beta)}{\Gamma(\beta)}\right) + C_3 T^{-1} \end{array}\right) \quad (53)$$

To find the minimum value of the total average cost $TOC^{av}_{\alpha,m,\beta}(T)$, we proposed primal geometric programming method as model-I, case-(i).

$$\text{Min}TOC^{av}_{\alpha,m,\beta}(T) = \begin{cases} AT^{(\alpha-1)} + B_1 T^{\alpha+m-1} + CT^{\alpha+\beta-1} + DT^{(\alpha+\beta+m-1)} + ET^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (54)$$

$$\left(\begin{array}{l} \text{Where, } A = \frac{aU}{\Gamma(1+\alpha)}, \quad B_1 = \frac{bU\Gamma(1+m)}{\Gamma(\alpha+m+1)}, \\ C = \frac{C_1 a}{\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\beta)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right), \\ D = \frac{C_1 b\Gamma(1+m)}{\Gamma(\alpha+m+1)}\left(\frac{1}{\Gamma(1+\beta)} - \frac{B(\alpha+m+1,\beta)}{\Gamma(\beta)}\right), \\ E = C_3 \end{array}\right)$$

(ii)Case-2: $m = 1.0\ , 0 < \alpha \le 1.0, 0 < \beta \le 1.0.$

Total average cost becomes as,

$$TOC_{\alpha,1,\beta}(T) = \frac{\left(UQ + HOC_{\alpha,1,\beta}(T) + C_3\right)}{T} = \left(\begin{array}{l} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bU\Gamma(2)T^{\alpha}}{\Gamma(\alpha+2)} \\ + \frac{C_1 aT^{(\alpha+\beta-1)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right) \\ + \frac{C_1 b\Gamma(2)T^{(\alpha+\beta)}}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right) \\ + C_3 T^{-1} \end{array}\right) \quad (55)$$

Then system (54) reduces to

$$\text{Min}TOC^{av}_{\alpha,1,\beta}(T) = \begin{cases} AT^{(\alpha-1)} + B_1 T^{\alpha} + CT^{\alpha+\beta-1} + DT^{(\alpha+\beta)} + ET^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (56)$$

$$\left(\begin{array}{l} \text{Where, } A = \frac{aU}{\Gamma(1+\alpha)}, B_1 = \frac{bU\Gamma(2)}{\Gamma(\alpha+2)}, \\ C = \frac{C_1 a}{\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\beta)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right), \\ D = \frac{C_1 b\Gamma(2)}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(1+\beta)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right), \\ E = C_3 \end{array}\right)$$

In the similar manner as in case (i) of model-I, $TOC^*_{\alpha,1,\beta}(T)$ and $T^*_{\alpha,1,\beta}$ can be found analytically.

$(\textbf{iii})\,\textbf{Case}-\textbf{3}: 0 < m \le 1.0, 0 < \alpha \le 1.0, \text{and } \beta = 1.0.$

In this case, total average cost is as,

$$TOC_{\alpha,m,1}(T) = \frac{\left(UQ + HOC_{\alpha,m,1}(T) + C_3\right)}{T} = \left(\begin{array}{l} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bU\Gamma(1+m)T^{\alpha+m-1}}{\Gamma(\alpha+m+1)} + \frac{C_1 aT^{(\alpha)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(2)} - \frac{B(\alpha+1,1)}{\Gamma(1)}\right) \\ + \frac{C_1 b\Gamma(1+m)T^{(\alpha+m)}}{\Gamma(\alpha+m+1)}\left(\frac{1}{\Gamma(2)} - \frac{B(\alpha+m+1,1)}{\Gamma(1)}\right) + C_3 T^{-1} \end{array}\right) \quad (57)$$

For this parametric values the system (54) reduces to

$$\text{Min}TOC^{av}_{\alpha,m,1}(T) = \begin{cases} AT^{(\alpha-1)} + B_1 T^{\alpha+m-1} + CT^{\alpha} + DT^{(\alpha+m)} + ET^{-1} \\ \text{Subject to } T \geq 0 \end{cases} \quad (58)$$

$$\left(\begin{array}{l} \text{Where, } A = \frac{aU}{\Gamma(1+\alpha)}, B_1 = \frac{bU\Gamma(1+m)}{\Gamma(\alpha+m+1)}, \\ C = \frac{C_1 a}{\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(2)} - \frac{B(\alpha+1,1)}{\Gamma(1)}\right), \\ D = \frac{C_1 b\Gamma(1+m)}{\Gamma(\alpha+m+1)}\left(\frac{1}{\Gamma(2)} - \frac{B(\alpha+m+1,1)}{\Gamma(1)}\right), \\ E = C_3 \end{array}\right)$$

In the similar way, as in case (i) of model-I, it helps to give the results of minimized total average cost and optimal ordering interval $TOC^{*}_{\alpha,m,1}(T)$ and $T^{*}_{\alpha,m,1}(T)$ respectively analytically.

(iv) Case-4: $\alpha = 1.0, 0 < m \leq 1.0, 0 < \beta \leq 1.0.$

In this case, total average cost is as,

$$TOC_{1,m,\beta}(T) = \frac{\left(UQ + HOC_{1,m,\beta}(T) + C_3\right)}{T} = \left(\begin{array}{l} \frac{aUT^{0}}{\Gamma(2)} + \frac{bU\Gamma(1+m)T^{m}}{\Gamma(m+2)} \\ + \frac{C_1 aT^{(\beta)}}{\Gamma(2)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(m+2,\beta)}{\Gamma(\beta)}\right) \\ + \frac{C_1 b\Gamma(2)T^{(m+\beta)}}{\Gamma(3)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(3,\beta)}{\Gamma(\beta)}\right) + C_3 T^{-1} \end{array}\right)$$

Then, the system (54) will reduce to

$$\text{Min}TOC^{av}_{1,m,\beta}(T) = \begin{cases} AT^{(0)} + B_1 T^{m} + CT^{\beta} + DT^{(m+\beta)} + ET^{-1} \\ \text{Subject to } T \geq 0 \end{cases} \quad (59)$$

$$\left(\begin{array}{l} \text{Where, } A = \frac{aU}{\Gamma(2)}, B_1 = \frac{bU\Gamma(1+m)}{\Gamma(m+2)}, \\ C = \frac{C_1 a}{\Gamma(2)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(2,\beta)}{\Gamma(\beta)}\right), \\ D = \frac{C_1 b\Gamma(2)}{\Gamma(3)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(m+2,\beta)}{\Gamma(\beta)}\right), \\ E = C_3 \end{array}\right) \quad (60)$$

Using primal geometric programming algorithm we can find minimized total average cost $TOC^{*}_{1,\text{m},\beta}$ and the optimal ordering interval $T^{*}_{1,\text{m},\beta}$ as describe in case-1 of model-I.

(v)Case-5: $m = \beta = 1.0$ and $0 < \alpha \leq 1.0.$

In this case, total average cost becomes as

$$TOC_{\alpha,1,1}(T) = \frac{\left(UQ + HOC_{\alpha,1,1}(T) + C_3\right)}{T} = \left(\begin{array}{l} \frac{aUT^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{bU\Gamma(2)T^{\alpha}}{\Gamma(\alpha+2)} \\ + \frac{C_1 aT^{(\alpha)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(2)} - \frac{B(\alpha+1,1)}{\Gamma(1)}\right) \\ + \frac{C_1 b\Gamma(2)T^{(\alpha+1)}}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(2)} - \frac{B(\alpha+2,1)}{\Gamma(1)}\right) \\ + C_3 T^{-1} \end{array}\right) \quad (61)$$

Therefore, generalized inventory model (54) reduces to,

$$\text{Min}TOC^{av}_{\alpha,1,1}(T) = \begin{cases} AT^{(\alpha-1)} + B_1 T^{\alpha} + CT^{\alpha+1} + DT^{(-1)} \\ \text{Subject to } T \geq 0 \end{cases} \quad (62)$$

$$\left(\begin{array}{l}\text{Where}, A=\frac{aU}{\Gamma(1+\alpha)},\\ B_1=\frac{bU\Gamma(2)}{\Gamma(\alpha+2)}+\frac{C_1a}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(2)}-\frac{B(\alpha+1,1)}{\Gamma(1)}\right),\\ C=\frac{C_1b\Gamma(2)T^{(\alpha+1)}}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(2)}-\frac{B(\alpha+2,1)}{\Gamma(1)}\right),\\ E=C_3\end{array}\right)$$

In the similar way as in the case (i) of model-I, analytically, primal geometric programming algorithm can give the minimized total average cost $TOC^*_{\alpha,1,1}$ and optimal ordering interval $T^*_{\alpha,1,1}$.

(vi) Case-6: $m=1.0, \alpha=1.0,\ 0<\beta\le 1.0$

In this case, total average cost is as,

$$TOC_{1,1,\beta}(T)=\frac{\left(UQ+HOC_{1,1,\beta}(T)+C_3\right)}{T}$$
$$=\left(\begin{array}{l}\frac{aUT^0}{\Gamma(2)}+\frac{bU\Gamma(2)T^1}{\Gamma(3)}+\frac{C_1aT^{(\beta)}}{\Gamma(2)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(2,\beta)}{\Gamma(\beta)}\right)\\ +\frac{C_1b\Gamma(2)T^{(1+\beta)}}{\Gamma(3)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(3,\beta)}{\Gamma(\beta)}\right)+C_3T^{-1}\end{array}\right) \quad (63)$$

Therefore, the generalized inventory model (54) reduces to,

$$\text{Min}TOC^{av}_{1,1,\beta}(T)=\begin{cases}AT^{(0)}+B_1T^1+CT^{\beta}+DT^{(\beta+1)}+ET^{(-1)}\\ \text{Subject to } T\ge 0\end{cases} \quad (64)$$

$$\left(\begin{array}{l}\text{Where}, A=\frac{aU}{\Gamma(2)}, B_1=\frac{bU\Gamma(2)}{\Gamma(3)},\\ C=\frac{C_1a}{\Gamma(2)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(2,\beta)}{\Gamma(\beta)}\right),\\ D=\frac{C_1b\Gamma(2)}{\Gamma(3)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(3,\beta)}{\Gamma(\beta)}\right),\\ E=C_3\end{array}\right)$$

In the similar manner as in case (i) of model-**I**, primal geometric programming algorithm gives the analytical results of the minimized total average cost $TOC^*_{1,1,\beta}$ and optimal ordering interval $T^*_{1,1,\beta}$.

(vii)Case-7: $\alpha=\beta=1.0$ and $0<m\le 1.0.$

In this case, total average cost becomes as,

$$TOC_{1,m,1}(T)=\frac{\left(UQ+HOC_{1,m,1}(T)+C_3\right)}{T}$$
$$=\left(\begin{array}{l}\frac{aUT^0}{\Gamma(2)}+\frac{bU\Gamma(1+m)T^m}{\Gamma(m+2)}+\frac{C_1aT^{(1)}}{\Gamma(2)}\left(\frac{1}{\Gamma(2)}-\frac{B(2,1)}{\Gamma(1)}\right)\\ +\frac{C_1b\Gamma(1+m)T^{(m+1)}}{\Gamma(m+2)}\left(\frac{1}{\Gamma(2)}-\frac{B(m+2,1)}{\Gamma(1)}\right)+C_3T^{-1}\end{array}\right) \quad (65)$$

In this case, the generalized inventory model (54) becomes as,

$$\text{Min}TOC^{av}_{1,m,1}(T)=\begin{cases}AT^{(0)}+B_1T^m+CT^1+DT^{(m+1)}+ET^{(-1)}\\ \text{Subject to } T\ge 0\end{cases} \quad (66)$$

In the similar manner as in case (i) of model-I, primal geometric programming algorithm can give the minimized total average cost $TOC^*_{1,m,1}$ and optimal ordering interval $T^*_{1,m,1}$.

(viii)Case-8: $\alpha=m=\beta=1.0.$

In this case, total average cost is as,

$$TOC_{1,1,1}(T)=\frac{\left(UQ+HOC_{1,1,1}(T)+C_3\right)}{T}$$
$$=\left(\begin{array}{l}\frac{aUT^0}{\Gamma(2)}+\frac{bU\Gamma(2)T^1}{\Gamma(3)}+\frac{C_1aT^{(1)}}{\Gamma(2)}\left(\frac{1}{\Gamma(2)}-\frac{B(2,1)}{\Gamma(1)}\right)\\ +\frac{C_1b\Gamma(2)T^{(2)}}{\Gamma(3)}\left(\frac{1}{\Gamma(2)}-\frac{B(3,1)}{\Gamma(1)}\right)+C_3T^{-1}\end{array}\right) \quad (67)$$

In this case, the generalized inventory model (54) becomes as,

$$\text{Min}TOC^{av}_{1,1,1}(T)=\begin{cases}AT^{(0)}+B_1T+CT^2+DT^{(-1)}\\ \text{Subject to } T\ge 0\end{cases} \quad (68)$$

$$\left(\begin{array}{l} A=\frac{aU}{\Gamma(2)}, B_1=\left(\begin{array}{l}\frac{bU\Gamma(2)}{\Gamma(3)}+\\ \frac{C_1aT^{(1)}}{\Gamma(2)}\left(\frac{1}{\Gamma(2)}-\frac{B(2,1)}{\Gamma(1)}\right)\end{array}\right),\\ C=\frac{C_1b\Gamma(2)T^{(2)}}{\Gamma(3)}\left(\frac{1}{\Gamma(2)}-\frac{B(3,1)}{\Gamma(1)}\right),\\ D=C_3 \end{array}\right)$$

The expression (68) coincides to the expression (11) for $\alpha=m=\beta=1.0.$

In the similar way as in case (i) of model-I, primal geometric programming algorithm can provide the minimum value of the total average cost $TOC^{*}_{1,1,1}$ and optimal ordering interval $T^{*}_{1,1,1}$.

4. NUMERICAL ILLUSTRATIONS

(i) To illustrate numerically the developed classical and fractional order inventory model we consider empirical values of the various parameters in proper units as $a=10, b=6, U=50, C_1=7, C_3=20$ .The optimal ordering interval, minimized total average cost of the classical inventory model is found 0.3211units and 623.1329 units respectively.

(ii) Here, we provide a numerical illustration for the fractional order inventory models considering same parameters as used in classical inventory model.

α	$T^{*}_{\alpha,\beta}$	$TOC^{*}_{\alpha,\beta}$
0.1	**4.0143**	589.5088
0.14360143	**3.6653**	**623.1329**
0.2	3.2621	664.5379
0.3	2.6519	729.9727
0.4	2.1417	781.5483
0.5	1.7046	815.0694
0.6	1.3240	**826.6948**
0.7	0.9905	813.3189
0.8	0.7035	773.2086
0.9↑(growing memory effect)	0.4739	707.4710
1.0	0.3211	**623.1329**

Table-2(a): Optimal ordering interval and minimized total average cost $TOC^{*}_{\alpha,\beta}$ for $\beta=1.0$, and α varies from 0.1 to 1.0 as defined in section 3.4.1(fractional model-**I**) case-**2**.

It is clear from the table-2(a) that there is a critical value of the memory parameter (here it is $\alpha=0.6$),for which the minimized total average cost becomes maximum and then gradually decreases bellow and above. For the above case, low values of α signifies large memory of the inventory problem. There is another critical value of memory parameter (here it is $\alpha=0.14360143$)for which minimum value of the total average cost becomes equal to the memory less minimized total average cost but optimal ordering interval is different. For large memory effect,(which is $\alpha=0.14360143$)the system needs more time to reach the minimum value of the total average cost compared to memory less system i.e. for the large memory effect, the system will take longer time to sell the optimum lot size compared to the memory less inventory system. Hence rate of selling decreases in large memory affected system. Therefore, to reach the same profit like memory less system, the shopkeeper should be changed his business policy such as attitude of public dealing, environment of shop or company, product quality etc. An observation has also been observed that there is some situation for which the system is very much attacked by the bad memory and then the company or shopkeeper has recovered his business policy. The above described facts happening in real life inventory system but we are not able to include in terms of classical model. Initially the business started with reputation with maximum profit minimizing the total average cost. As time goes on, the company starts to lose it's reputation due to the various unwanted causes. Accordingly, the company starts to downfall of its business when downfall becomes maximum at $\alpha=0.6$ .Attaining highest value at that point, the company changes its business policy and takes care to recover its reputation.

α	β	$T^{*}_{\alpha,\beta}$	$TOC^{*}_{\alpha,\beta}$
1.0	0.1	0.3596	616.5578
1.0	0.2	0.3522	621.5138
1.0	**0.244157**	**0.3487**	**623.1329**
1.0	0.3	0.3443	624.7429

1.0	0.4	0.3371	626.5758
1.0	**0.5**	**0.3311**	**627.3265**
1.0	0.6	0.3265	627.2755
1.0	0.7	0.3234	626.6604
1.0	0.8	0.3216	625.6723
1.0	0.9↑(growing memory effect)	0.3209	624.4596
1.0	**1.0**	**0.3211**	**623.1329**

Table-2(b): Optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\beta}$ for $\alpha = 1.0$, and β varies from 0.1 to 1.0 as defined in section 3.4.1(fractional model-**I**) case-**3**.

It is clear from the table-2(b), that there is a critical memory value of the memory parameter β (here it is $(\beta = 0.5)$) for which minimized total average cost $TOC^*_{1,\beta}$ becomes **maximum** and then gradually decreases below and above. There is also clear that for the memory value $\beta = 0.244157$ for which minimized total average cost becomes equal to the memory less minimized total average cost but optimal ordering interval keeps small difference between them. Hence, for the large memory effect corresponding β, the system does not take significantly more time to reach the minimum value of the total average cost compared to the memory less system. Practically, β is the memory parameter corresponding inventory holding cost or carrying cost. Here, memory or past experience is considered as bad or good attitude of the shopkeeper to the transportation driver. But, in general the transportation driver does not react corresponding bad attitude of the shopkeeper. On the other hand, transportation driver may be bad as a service man i.e. he may not serious for his duty. Due to the above reason, the system is affected by the poor service of the transportation but this is not effective too which is also proved from the table-2(b).

In table-2(b), $TOC^*_{\alpha,\beta}$ does not show similar behavior as in table-2(a) without the pt ($\alpha = 1.0$ and $\beta = 1.0$ and in this case the ordering interval $T^*_{\alpha,\beta}$ is less i.e for $\alpha = 1.0$ and fractionally varying β lesser time is required to reach the minimum value of the total average cost.

α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	10.6778	473.2964
0.2	6.3706	566.1366
0.27247464	**4.7926**	**626.5758**
0.3	4.3480	647.7863
0.4	3.1396	715.2515
0.5	2.3113	764.6220
0.6	1.6926	791.7361
0.7	**1.2062**	**792.6978**
0.8	0.8178	764.5842
0.9↑(growing memory effect)	0.5233	707.0966
1.0	**0.3371**	**626.5758**

Table-2(c): Optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\beta}$ for $\beta = 0.4$, and α varies from 0.1 to 1.0 as defined in section 3.4.1(fractional model-**I**) case-**1**. (↑ symbolizes for increasing gradually)

It is clear from table-2(c) that for the critical memory value (here it is $\alpha = 0.7, \beta = 0.4$) where minimized total average cost becomes maximum and then gradually decreases below and above.

When exponent of holding cost is fractional $\beta = 0.4$ and another memory parameter (α) varies, minimized total average cost $TOC_{0.27247464,0.4}$ is same to the minimized total average cost $TOC_{1.0,0.4}$ but in the optimal ordering interval has difference. For the large memory effect of the memory parameter (α), system takes more time to reach the minimum value of the total average cost compared to the low memory effect.

In table-2(c), the minimized total average cost $\left(TOC^*_{\alpha,\beta}\right)$ shows similar behavior as in table-2(a) but the optimal ordering interval does not behave similarly as in table-2(a).

The following table-3(a) and table-3(b) has been constructed for the model-II, where rate of change of inventory level is of fractional order α and the highest degree of the demand polynomial is also fractional order α .

For $\alpha = 1.0$, and β varies from 0.1 to 1.0 the model-II and model-I are identical. The same numerical results have been displayed in the table-2(b).

α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	**286.1428**	**40.9294**
0.2	62.6096	140.1892
0.3	23.6949	298.7682
0.4	10.7735	492.2980
0.5	5.2222	673.9411
0.6	2.6011	794.6458
0.7	**1.3588**	**831.5622**
0.8	0.7672	796.8806
0.9↑(growing memory effect)	0.4716	718.7286
1.0	0.3211	623.1329

Table-3(a): Optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\beta}$ for $\beta = 1.0$ and α varies from 0.1 to 1.0 as defined in section 3.4.2(fractional model-**II**) case-**2**. (↑ uses for increasing gradually)

It is obvious from table-3(a), that the minimized total average cost $TOC^*_{\alpha,\beta}$ (where $\beta = 1.0$) is **maximum** at the memory parameter $\alpha = 0.7$ and gradually decreases below and above. Since for $\beta = 0.4$ the value of the minimized total average cost $TOC^*_{\alpha,\beta}$ is maximum at $\alpha = 0.7$ in model-II but it attains maximum at $\alpha = 0.6$ in model-**I**. This fact happens for consideration of fractional polynomials in demand rate in the model-**II**. For the demurrage of inventory for long of the optimal ordering interval there is less significance of the optimal ordering interval $T^*_{0.1,1}, T^*_{0.2,1}$ .In this case the model will realistic on or after $\alpha = 0.3$.

α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	**1.0000x10^4**	**0.4783**
0.2	1.0000x10^4	2.9736
0.3	1.0000x10^4	18.9921
0.4	5.5094x10^3	119.8284
0.5	96.9713	408.3475
0.6	8.0923	791.7361
0.7	**0.9570**	**792.6978**
0.8	0.5253	764.5842
0.9↑(growing memory effect)	0.5233	707.0966
1.0	**0.3371**	**626.5758**

Table-3(b): Optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\beta}$ for $\beta = 0.4$ and α varies from 0.1 to 1.0 as defined in section 3.4.1(fractional model-**II**) case-**1**. (↑ uses for increasing gradually)

From table-3(b) it is clear that for large memory effect (here it is $\alpha = 0.1, \beta = 0.4$) the system takes more time to reach the minimum value of the total average cost compared to the low memory effect(here it is $\alpha = 1.0, \beta = 0.4$) .Hence, the business stay long time to reach the minimum value of the total average cost. In this case, the model will realistic on or after $\alpha = 0.6$ and $\beta = 0.4$ .Hence, in this case, short memory has been observed compared to the other case.

The following table-4(a) has been constructed for model-III, where rate of change of inventory level is of fractional order α and the highest degree of the demand polynomial is also fractional *m*.

For $\beta = m = 1.0$ and α varies from 0.1 to 1.0 in fractional model-III (described in section 3.4.3, case-**v**) coincides with model-I (case-**2**).The obtained numerical results are same as given in table-2(b)

α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	**1.0000x10^4**	**6.0313**
0.2	1.0000x10^4	16.7221
0.3	1.0000x10^4	45.5166
0.4	5.5094x10^3	120.7342
0.5	96.9713	263.2125
0.6	8.0923	458.6571
0.7	0.9570	659.0075
0.8	0.5253	792.5776
0.9↑(growing memory effect)	0.5233	**803.6163**

1.0	**0.3371**	**718.8122**

Table-4(a): Optimal ordering interval and minimized total average cost $TOC^{*}_{\alpha,\mathrm{m},\beta}$ for $\beta = m = 0.4$ and α varies from 0.1 to 1.0 as defined in section 3.4.3(fractional model-**III**) case-**1**.(↑ uses for increasing gradually)

From the table-4(a), it is clear that for large memory effect (here it is $\alpha = 0.1, \beta = m = 0.4)$ the system takes more time to reach the minimum value of the total average cost compared to the low memory effect(here it is $\alpha = 1.0, \beta = m = 0.4)$ .Hence, for large memory effect, the business stay long time to reach the minimum value of the total average cost. Here also short memory has worked as in the table-3(a) and table-3(b).

Now, if we consider $\alpha = m = 1.0$ and β varies from 0.1 to 1.0 in the fractional model-III (described in section 3.4.3, case-vi) coincides with model-**I** (case-**3**).The obtained numerical results are same as given in table-2(b).

5. CONCLUSIONS

The purpose of the article is to describe different type generalization of an inventory model to take into account memory effect via fractional calculus. An observation has been found that for all situations of all fractional order inventory model, there is a critical memory effect for which minimized total average cost is maximum and then gradually decreases bellow and above. This critical memory effect indicates that in that situation, profit of the system is low for poor memory effect or poor experiences. The memory parameter α plays major role to take into account memory of the system compared to the another memory parameter β. Due to consider fractional type demand rate, short term memory has been observed which is appropriate for newly started business. For long memory affected business, our first fractional model is suitable. More work with vision needs to be carried out for future aspects.

ACKNOWLEDGEMENTS

The authors would like to thank the reviewers and the editor for the valuable comments and suggestions. The work is also supported by Prof R.N.Mukerjee, Retired professor of Burdwan University. The authors also would like to thank the Department of Science and Technology, Government of India, New Delhi, for the financial assistance under AORC, Inspire fellowship Scheme towards this research work.

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