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Chromatic Excellence in Hesitancy Fuzzy Graphs

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Abstract- Let G be a simple hesitancy fuzzy graph. A family $C = \{c_1, \ldots, c_k\}$ of hesitancy fuzzy sets on a set V is called a k-vertex coloring of $G = (V, E)$ if

(i) $\vee\ c_i(x) = V$, for all $x \in V$

(ii) $c_i \wedge c_j = 0$

(iii) For every strong edge xy of G, $\min\{c_i(\mu_1(x)), c_i(\mu_1(y))\} = 0$, $\max\{c_i(\gamma_1(x)), c_i(\gamma_1(y))\} = 1$ and $\min\{c_i(\beta_1(x)), c_i(\beta_1(y))\} = 0$, $(1 \le i \le k)$

The least value of k for which the G has a k-vertex coloring denoted by $\chi(G)$, is called the chromatic number of the hesitancy fuzzy graph G. Then C is the partition of independent sets of vertices of G in which each set has the same color is called the chromatic partition. A hesitancy fuzzy graph G is called the χ-excellent if every vertex of G appears as a singleton in some χ-partitions of G. The focal point of this paper is to study the new concept called chromatic excellence in hesitancy fuzzy graphs. Hesitancy fuzzy corona and hesitancy fuzzy independent sets are defined and studied. We explain these new concepts through illustrative examples.

Keywords- Chromatic number, Chromatic excellence, Hesitancy fuzzy graph, Vertex coloring.
AMS Subject Classification (2010)- 05C72, 05C15.

1. INTRODUCTION

Graph coloring dates back to 1852, when Francis Guthrie come up with the four color conjecture. Gary Chartrand and Ping Zhang [4] discussed various colorings of graph and its properties in their book entitled Chromatic Graph Theory. A graph coloring is the assignment of a color to each of the vertices or edges or both in such a way that no two adjacent vertices and incident edges share the same color. E. Sambathkumar discussed chromatically fixed, free and totally free vertices of a graph in 1992 [17]. Graph coloring has been applied to many real world problems like scheduling, allocation, telecommunications and bioinformatics, etc.

The concept of fuzzy sets and fuzzy relations were introduced by L.A.Zadeh in 1965 [22]. A. Rosenfeld who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975 [16]. The concept of chromatic number of fuzzy graph was introduced by Munoz et.al. in 2004 [6]. C. Eslahchi and B.N. Onagh introduced fuzzy graph coloring of fuzzy graph in 2006 [2]. S. Lavanya and R. Sattanathan discussed total fuzzy coloring in 2009 [5]. Anjaly Kishore and M.S. Sunitha discussed chromatic number of fuzzy graph in 2013 [1]. A. Nagoor Gani and B. Fathima Kani deliberated Fuzzy vertex order colouring in 2016 [8]. K.M. Dharmalingam and R. Udaya Suriya conferred chromatic excellence in fuzzy graph in 2017 [3].

R. Seethalakshmi and R.B. Gnanajothi introduced Anti-fuzzy graph (AFG) in 2016 [18] and discussed various properties in 2017 [19]. R. Muthuraj and A. Sasireka discussed anti- fuzzy graphs in 2017 [7]. A. Prasanna et.al. deliberated anti fuzzy graph coloring in 2018 [14] and conferred chromatic excellence in anti-fuzzy graph in 2018 [15].

Hesitant fuzzy sets introduced by V. Torra in 2010 [20]. T. Pathinathan et.al. introduced Hesitancy fuzzy graph in 2015 [9] and discussed various properties in [10, 11, 12]. N. Vinothkumar and G. Geetharamani discussed operations in hesitancy fuzzy graphs in 2018 [21]. A. Prasanna et.al. deliberated hesitancy fuzzy graph coloring in 2018 [13]. Hence in this paper we are introducing a novel concept called chromatic excellence in hesitancy fuzzy graph and its properties.

2. PRELIMINARIES

2.1. Definition *(L.A. Zadeh [22])*

Let X be a non-empty set. Then a fuzzy set A in X (i.e., a fuzzy subset A of X) is characterized by a function of the form $\mu_A: X \to [0,1]$, such a function μ_A is called the membership function and for each $x \in X$, $\mu_A(x)$ is the degree of membership of x (membership grade of x) in the fuzzy set A. In otherwords, $A = \{(x, \mu_A(x))/\ x \in X\}$ where $\mu_A: X \to [0,1]$.

2.2. Definition *(A. Rosenfeld [16])*

A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$, where for all $u, v \in V$, we have $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$.

2.3. Definition *(T. Pathinathan et.al [9])*

Hesitancy Fuzzy Graph (HFG) is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\mu_1: V \to [0,1]$, $\gamma_1: V \to [0,1]$ and $\beta_1: V \to [0,1]$ denote the degrees of membership, non-membership and hesitancy of the element $v_i \in V$ respectively and $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$, for every $v_i \in V$, $(i = 1,2,\dots,n)$, where $\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)]$, and $0 \le \mu_1(v_i) + \gamma_1(v_i) \le 1$.

(ii) $E \subseteq V \times V$ where $\mu_2: V \times V \to [0,1]$, $\gamma_2: V \times V \to [0,1]$ and $\beta_2: V \times V \to [0,1]$ are such that,

$$\mu_2(v_i, v_j) \le \min[\mu_1(v_i), \mu_1(v_j)]$$
$$\gamma_2(v_i, v_j) \le \max[\gamma_1(v_i), \gamma_1(v_j)]$$
$$\beta_2(v_i, v_j) \le \min[\beta_1(v_i), \beta_1(v_j)]$$

And $0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) + \beta_2(v_i, v_j) \le 1$ for every $(v_i, v_j) \in E$.

Here the $(v_i, \mu_{1i}, \gamma_{1i}, \beta_{1i})$ denote the vertex, the degree of membership, degree of non-membership and hesitancy of the vertex v_i. And the $(e_{ij}, \mu_{2ij}, \gamma_{2ij}, \beta_{2ij})$ denote the edge, the degree of membership, degree of non-membership and hesitancy of the edge relation $e_{ij} = (v_i, v_j)$ on $V \times V$.

2.4. Definition *(K.M. Dharmalingam et.al [3])*

G is fuzzy chromatic excellent if for every vertex of $v \in V(G)$, there exists a fuzzy chromatic partitions Γ^f such that $\{v\} \in \Gamma^f$.

2.5. Definition *(R.Seethalakshmi et.al [18])*

An anti-fuzzy graph (AFG) $G = (\sigma, \mu)$ is a pair of functions $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$, where for all $u, v \in V$, we have $\mu(u,v) \ge \sigma(u) \vee \sigma(v)$.

2.6. Definition *(A. Prasanna et.al [15])*

A graph G is an anti-fuzzy chromatic excellent if for every vertex of $v \in V(G)$, there exists an anti-fuzzy chromatic partition C such that $\{v\} \in C$.

2.7. Definition *(N. Vinothkumar et.al [21])*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are hesitancy fuzzy graphs with $V_1 \cap V_2 = \emptyset$. The union of G_1 and G_2 is a hesitancy fuzzy graph on $V_1 \cup V_2$ and it is defined by
$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

In $V_1 \cup V_2$,

$$(\mu_{11} \cup \mu_{21})(u) = \begin{cases} \mu_{11}(u) & if\ u \in V_1 \\ \mu_{21}(u) & if\ u \in V_2 \end{cases}$$
$$(\gamma_{11} \cup \gamma_{21})(u) = \begin{cases} \gamma_{11}(u) & if\ u \in V_1 \\ \gamma_{21}(u) & if\ u \in V_2 \end{cases}$$
$$(\beta_{11} \cup \beta_{21})(u) = \begin{cases} \beta_{11}(u) & if\ u \in V_1 \\ \beta_{21}(u) & if\ u \in V_2 \end{cases}$$

and in $E_1 \cup E_2$,

$$(\mu_{12} \cup \mu_{22})(uv) = \begin{cases} \mu_{12}(uv) & if\ uv \in E_1 \\ \mu_{22}(uv) & if\ uv \in E_2 \end{cases}$$
$$(\gamma_{12} \cup \gamma_{22})(uv) = \begin{cases} \gamma_{12}(uv) & if\ uv \in E_1 \\ \gamma_{22}(uv) & if\ uv \in E_2 \end{cases}$$
$$(\beta_{12} \cup \beta_{22})(uv) = \begin{cases} \beta_{12}(uv) & if\ uv \in E_1 \\ \beta_{22}(uv) & if\ uv \in E_2 \end{cases}$$

2.8. Definition *(N. Vinothkumar et.al [21])*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are hesitancy fuzzy graphs with $V_1 \cap V_2 = \emptyset$. The join of G_1 and G_2 is a hesitancy fuzzy graph on $V_1 \cup V_2$ and it is defined by $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ where E' is the set of all edges joining the vertices of V_1 and V_2.

In $V_1 \cup V_2$, $(\mu_{11} + \mu_{21})(u) = (\mu_{11} \cup \mu_{21})(u)$, $(\gamma_{11} + \gamma_{21})(u) = (\gamma_{11} \cup \gamma_{21})(u)$, $(\beta_{11} + \beta_{21})(u) = (\beta_{11} \cup \beta_{21})(u)$,

And in $E_1 \cup E_2$,

$$(\mu_{12} + \mu_{22})(uv) = \begin{cases} (\mu_{12} \cup \mu_{22})(uv) & if\ (uv) \in E_1 \cup E_2 \\ \min\{\mu_{11}(u), \mu_{21}(v)\} & if\ (uv) \in E' \end{cases}$$
$$(\gamma_{12} + \gamma_{22})(uv) = \begin{cases} (\gamma_{12} \cup \gamma_{22})(uv) & if\ (uv) \in E_1 \cup E_2 \\ \max\{\gamma_{11}(u), \gamma_{21}(v)\} & if\ (uv) \in E' \end{cases}$$
$$(\beta_{12} + \beta_{22})(uv) = \begin{cases} (\beta_{12} \cup \beta_{22})(uv) & if\ (uv) \in E_1 \cup E_2 \\ \min\{\beta_{11}(u), \beta_{21}(v)\} & if\ (uv) \in E' \end{cases}$$

2.9. Definition *(N. Vinothkumar et.al [21])*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are hesitancy fuzzy graphs with $V_1 \cap V_2 = \emptyset$. The Cartesian product of G_1 and G_2 is a hesitancy fuzzy graph on $V_1 \times V_2$ and it is defined by $G_1 \times G_2 = (V, E)$.

In vertex set V, $(\mu_{11} \times \mu_{21})(uv) = \mu_{11}(u) \wedge \mu_{21}(v)$, $(\gamma_{11} \times \gamma_{21})(uv) = \gamma_{11}(u) \vee \gamma_{21}(v)$, $(\beta_{11} \times \beta_{21})(uv) = \beta_{11}(u) \wedge \beta_{21}(v)$.

And in edge set E,

$$(\mu_{12} \times \mu_{22})((u_1u_2)\ (v_1v_2)) = \begin{cases} \mu_{11}(u_1) \wedge \mu_{22}(u_2v_2), if\ u_1 = v_1 \\ \mu_{21}(u_2) \wedge \mu_{12}(u_1v_1), if\ u_2 = v_2 \\ 0, \quad otherwise \end{cases}$$

$$(\gamma_{12} \times \gamma_{22})((u_1u_2)\ (v_1v_2)) = \begin{cases} \gamma_{11}(u_1) \vee \gamma_{22}(u_2v_2), if\ u_1 = v_1 \\ \gamma_{21}(u_2) \vee \gamma_{12}(u_1v_1), if\ u_2 = v_2 \\ 0, \quad otherwise \end{cases}$$

$$(\beta_{12} \times \beta_{22})\big((u_1u_2)\ (v_1v_2)\big) = \begin{cases} \beta_{11}(u_1) \wedge \beta_{22}(u_2v_2), if\ u_1 = v_1 \\ \beta_{21}(u_2) \wedge \beta_{12}(u_1v_1), if\ u_2 = v_2 \\ 0, \quad otherwise \end{cases}$$

3. MAIN DEFINITIONS AND RESULTS

3.1. Definition

Let G be a hesitancy fuzzy graph. A subset S of G is said to be a hesitancy fuzzy independent set of G, if there exists no $v_iv_j \in S$ such that $\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)]$, $\gamma_2(v_i, v_j) \leq \max[\gamma_1(v_i), \gamma_1(v_j)]$ and $\beta_2(v_i, v_j) \leq \min[\beta_1(v_i), \beta_1(v_j)]$. The maximum cardinality of such hesitancy fuzzy independent set is called a hesitancy fuzzy independence number and it is denoted by α_0.

3.2. Definition *(A. Prasanna et.al. [13])*

Let G be a hesitancy fuzzy graph. A family $C = \{c_1, \dots, c_k\}$ of hesitancy fuzzy sets on a set V is called a k-vertex coloring of $G = (V, E)$ if

(i) $\vee\ c_i(x) = V$, for all $x \in V$
(ii) $c_i \wedge c_j = 0$
(iii) For every strong edge xy of G, $\min\{c_i(\mu_1(x)), c_i(\mu_1(y))\} = 0$, $\max\{c_i(\gamma_1(x)), c_i(\gamma_1(y))\} = 1$ and $\min\{c_i(\beta_1(x)), c_i(\beta_1(y))\} = 0$, $(1 \leq i \leq k)$

The least value of k for which the G has a k-vertex coloring denoted by $\chi(G)$, is called the chromatic number of the hesitancy fuzzy graph G.

3.3. Definition

C is the partition of independent sets of vertices of a hesitancy fuzzy graph G in which each set has the same color is called the hesitancy fuzzy chromatic partition.

3.4. Definition

A vertex $v \in V(G)$ is called χ- good if $\{v\}$ belongs to some C –partition. Otherwise v is said to be C –bad vertex.

3.5. Definition

A graph is called χ-excellent hesitancy fuzzy graph if every vertex of G is χ-good.

3.6. Definition

A graph G is said to be C-commendable hesitancy fuzzy graph if the number of C-good vertices is greater than the number of C-bad vertices.

A graph G is said to be C-fair hesitancy fuzzy graph if the number of C-good vertices is equal to the number of C-bad vertices.

A graph G is said to be C-poor hesitancy fuzzy graph if the number of C-good vertices is less than the number of C-bad vertices.

3.7. Definition

Let $\{c_1, \dots, c_k\}$ be a χ-partitions of G. Let $v \in V$, then

(i) v is χ-fixed if $\{v\} \in C_i\ for\ all\ i, (1 \leq i \leq k)$
(ii) v is χ-free if for some $i, j, i \neq j$, $\{v\} \in C_i$ and $\{v\} \notin C_j$
(iii) v is χ-totally free if $\{v\} \notin C_i\ for\ all\ i, (1 \leq i \leq k)$.

3.8. Definition

A graph G is a hesitancy fuzzy chromatic excellent if for every vertex of $v \in V(G)$, there exists a hesitancy fuzzy chromatic partition C such that $\{v\} \in C$.

3.9. Remark

(i) Complete hesitancy fuzzy graph K_n is χ-excellent.
(ii) Hesitancy fuzzy cyclic graph C_{2n} is not χ-excellent but $C_{2n+1} (n \geq 1)$ is χ-excellent.
(iii) Hesitancy fuzzy wheel graph $W_{2n} (n \geq 2)$ is χ-excellent.

3.10. Example *(Hesitancy fuzzy graph vertex coloring)*

Consider the hesitancy fuzzy graph $G = (V, E)$ in figure 3.1.

Let $C = \{c_1, c_2, c_3, c_4\}$ be a family of hesitancy fuzzy sets defined on V as follows

$$c_1(u_i) = \begin{cases} (0.3,0.4,0.2),\ i = 1 \\ (0.4,0.3,0.2),\ i = 4 \\ (0),\ otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.6,0.2,0.1),\ i = 3 \\ (0.2,0.5,0.3),\ i = 5 \\ (0),\ otherwise \end{cases}$$

$$c_2(u_i) = \begin{cases} (0.5,0.1,0.4),\ i = 2 \\ (0),\ otherwise \end{cases}$$

$$c_4(u_i) = \begin{cases} (0.1,0.4,0.2),\ i = 6 \\ (0),\ otherwise \end{cases}$$

Hence $\chi(G) = 4$. The hesitancy-fuzzy chromatic partitions are

$$C_1 = \{\{v_2\}, \{v_6\}, \{v_1, v_4\}, \{v_3, v_5\}\}$$
$$C_2 = \{\{v_1\}, \{v_6\}, \{v_2, v_4\}, \{v_3, v_5\}\}$$
$$C_3 = \{\{v_3\}, \{v_6\}, \{v_1, v_4\}, \{v_2, v_5\}\}$$
$$C_4 = \{\{v_4\}, \{v_6\}, \{v_1, v_3\}, \{v_2, v_5\}\}$$
$$C_5 = \{\{v_5\}, \{v_6\}, \{v_1, v_3\}, \{v_2, v_4\}\}$$

Every vertex of G is appears in singleton in χ-partition, hence above graph (figure 3.1) is χ-excellent.

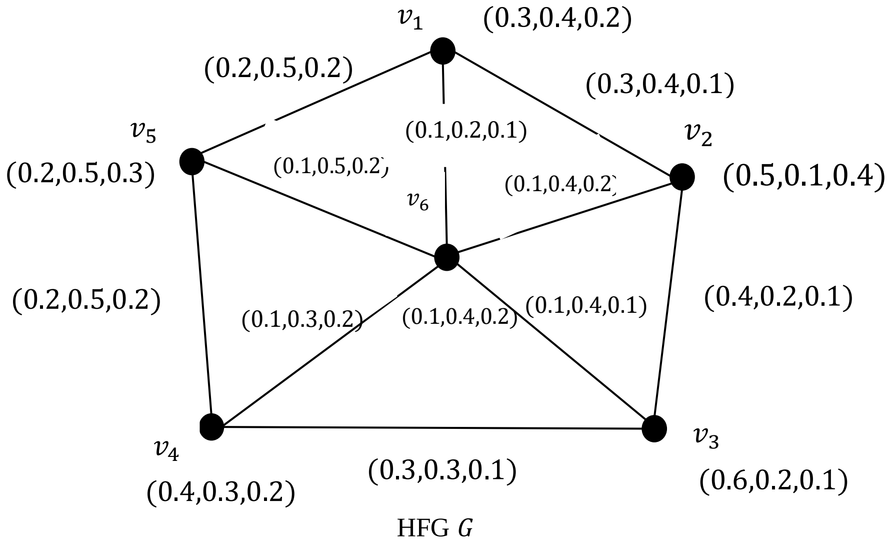

Figure 3.1

3.11. Definition

A hesitancy fuzzy graph G is α_0-excellent if every vertex belongs to a maximum hesitancy fuzzy independent set.

3.12. Remark

α_0-excellence and χ-excellence are unrelated properties. That is, a graph may be α_0-excellent but not χ-excellent and conversely.

3.13. Remark

(i) P_{2n} is not χ-excellent but not α_0-excellent.

(ii) $K_{1,n}$ is neither χ-excellent and α_0-excellent.

(iii) A graph in figure 3.1 is χ-excellent but not α_0-excellent.

3.14. Definition

Let $G_1$and G_2 be two hesitancy fuzzy graphs. Then the hesitancy fuzzy corona $G_1 \circ G_2$ is denoted by G^+ as the hesitancy fuzzy graph if taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining i^{th} vertex of G_1 to every vertex in i^{th} copy of G_2 such that $\mu_2(u_i, v_i) \leq \min[\mu_1(u_i), \mu_1(v_i)]$, $\gamma_2(u_i, v_i) \leq \max[\gamma_1(u_i), \gamma_1(v_i)]$ and $\beta_2(u_i, v_i) \leq \min[\beta_1(u_i), \beta_1(v_i)]$ for all $u_i \in i^{th}$ copy of G_2. If $G_2 = K_1$ then $G_1 \circ K_1$ is denoted by G_1^+.

3.15. Definition

Let G be a hesitancy fuzzy graph. The Mycielskian of G is the hesitancy fuzzy graph $\mu^h(G)$ with vertex set equal to $V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$ and the edge set $E \cup \{xy', x'y : xy \in E\}$ and $\mu_2(xy') \leq \min[\mu_1(x), \mu_1(y')]$, $\gamma_2(xy') \leq \max[\gamma_1(x), \gamma_1(y')]$, $\beta_2(xy') \leq \min[\beta_1(x), \beta_1(y')]$ and $\mu_2(x'y) \leq \min[\mu_1(x'), \mu_1(y)]$, $\gamma_2(x'y) \leq \max[\gamma_1(x'), \gamma_1(y)]$, $\beta_2(x'y) \leq \min[\beta_1(x'), \beta_1(y)]$. The vertex x' is called the twin of the vertex and the vertex u is called the root of $\mu^h(G)$.

3.16. Theorem

Let $\chi(G) = 2$ (then G is a bipartite hesitancy fuzzy graph). Let $|V(G)| \geq 3$ then G is not χ-excellent.

Proof.

Let us assume that G is χ-excellent hesitancy fuzzy graph. Since $\chi(G) = 2$, then there exists a chromatic partition $C = \{V_1, V_2\}$ of G such that $V_1 = \{v\}$.

Therefore $\langle V_2 \rangle = \langle G - \{v\} \rangle$ is totally disconnected graph. Clearly, $|V_2| \geq 2$ and $v \in V_1$ is adjacent to some vertex of V_2 such that $\mu_2(vu) \leq \min[\mu_1(v), \mu_1(u)]$, $\gamma_2(vu) \leq \max[\gamma_1(v), \gamma_1(u)]$, $\beta_2(vu) \leq \min[\beta_1(v), \beta_1(u)]$ for some $u \in V_2$.

Let $w \in V_2$ such that $wv \in E$ such that $\mu_2(vw) \leq \min[\mu_1(v), \mu_1(w)]$, $\gamma_2(vw) \leq \max[\gamma_1(v), \gamma_1(w)]$, $\beta_2(vw) \leq \min[\beta_1(v), \beta_1(w)]$. Let $w_1 \neq w \in V_2$. Since G is χ-excellent and $\chi(G) = 2$, then there exists a chromatic partition $C_1 = \{\{w_1\}, V_3\}$. Therefore $v, w \in V_3$ which is a contradiction to $vw \in E(G)$. Hence G is not χ-excellent.

3.17. Theorem

χ-excellent graphs has no hesitancy fuzzy isolates.

Proof.

Suppose G is a χ-excellent graph, which has an isolate v. If $G = K_n (n \geq 2)$, then $\chi(G) = 1$ and no vertex in G appears as a singleton in χ-partition of G. Therefore $G = K_n$. Hence $\chi(G) \geq 2$, then there exists a chromatic partition $C = \left\{\{v\}, V_2, V_3, \dots, V_{\chi}(G)\right\}$. Therefore $\deg(v) \geq \chi(G) - 1 \geq 1$. But v is an isolate, which is a contradiction. Hence any χ-excellent graph has no hesitancy fuzzy isolates.

3.18. Remark

If G is χ-excellent and $G \neq K_2$, then $\delta(G) \geq 2$.

3.19. Theorem

Let G_1 and G_2 be two hesitancy fuzzy graphs. Then $G_1 \cup G_2$ is not χ-excellent.

Proof.

If $V(G_1)\ \left(or\ V(G_2)\right)$ is a singleton, then clearly $G_1 \cup G_2$ is not χ-excellent. Let $|V(G_1)| \geq 2$, $|V(G_2)| \geq 2$.

Case (i). Let $\chi(G_1) = \chi(G_2) = k$

Suppose there exists a χ-partition of $G_1 \cup G_2$ such that $\{v\}$ is an element in the χ-partition for some $v \in V(G_1)$. Let $C = \left\{\{v\}, V_2, V_3, \dots, V_k\right\}$ be a χ-partition of $G_1 \cup G_2$. Therefore $\{V_2 - V(G_1), V_3 - V(G_1), \dots, V_k - V(G_1)\}$ be a proper color partition of G_2, and $\chi(G_2) \leq k - 1$, which is a contradiction that $\chi(G_2) = k$.

Similarly, we can show that there is no vertex $\{v\} \in V(G_2)$, can not appear as any χ-partition of $G_1 \cup G_2$. Hence $G_1 \cup G_2$ is not χ-excellent.

Case (ii). Let $\chi(G_1) \neq \chi(G_2)$. Let us assume that $\chi(G_2) = k$, then $\chi(G_1) \cup \chi(G_2) = k$. Suppose there exists a χ-partition of $G_1 \cup G_2$ such that $\{v\}$ is an element of the χ-partition for some $v \in V(G_1)$. Let $C = \left\{\{v\}, V_2, V_3, \dots, V_k\right\}$ be a χ-partition of $G_1 \cup G_2$. Then $\{V_2 - V(G_1), V_3 - V(G_1), \dots, V_k - V(G_1)\}$ be a proper color χ-partition of G_2 and hence $\chi(G_2) \leq k - 1$, which is a contradiction. Therefore $G_1 \cup G_2$ is not χ-excellent.

3.20. Corollary

If G is χ-excellent then G is a connected hesitancy fuzzy graph.

3.21. Remark

If G_1 and G_2 have same chromatic number, then no vertex of $G_1 \cup G_2$ can appear as a singleton in any χ-partition of $G_1 \cup G_2$.

If $\chi(G_1) \leq \chi(G_2)$, then we know that no vertex in G_1 can appear as a singleton in χ-partition of $G_1 \cup G_2$. But a vertex of G_2 may appear as a singleton in any χ-partition of $G_1 \cup G_2$.

3.22. Remark

$P_n, (n \geq 3)$ is not χ-excellent but it is an induced subgraph of an odd cycle which is χ-excellent. P_n is an induced subgraph of C_{n+1} if n is even and C_{n+2} if n is odd.

3.23. Theorem

$K_{1,n}$ is not χ-excellent but it is an induced subgraph of a χ-excellent graph.

Proof.

Let $V(K_{1,n}) = \{u, u_1, u_2, \dots, u_n\}$ where $u_1, u_2, \dots, u_n$ are independent (i.e., there is no edge $\mu_2(u_i, u_j) \leq \min[\mu_1(u_i), \mu_1(u_j)]$, $\gamma_2(u_i, u_j) \leq \max[\gamma_1(u_i), \gamma_1(u_j)]$ and $\beta_2(u_i, u_j) \leq \min[\beta_1(u_i), \beta_1(u_j)]$. Add vertices $v_1, v_2, \dots, v_n$ to $K_{1,n}$ such that $\mu_2(v_i, v_j) \leq \min[\mu_1(v_i), \mu_1(v_j)]$, $\gamma_2(v_i, v_j) \leq \max[\gamma_1(v_i), \gamma_1(v_j)]$ and $\beta_2(v_i, v_j) \leq \min[\beta_1(v_i), \beta_1(v_j)]$ for all $i \neq j, 1 \leq j \leq n$, and also $\mu_2(v_i, u_k) \leq \min[\mu_1(v_i), \mu_1(u_k)]$, $\gamma_2(v_i, u_k) \leq \max[\gamma_1(v_i), \gamma_1(u_k)]$ and $\beta_2(v_i, u_k) \leq \min[\beta_1(v_i), \beta_1(u_k)]$ for all $i \neq k, 1 \leq k \leq n$. Let G be the resulting graph with $\delta(G) = n$. Since G contains K_n, then $\chi(G) \geq n$. Suppose let $\chi(G) = n$ and $C = \{V_1, V_2, \dots, V_n\}$ be a χ-partition of G. Without loss of generality, $v_i \in V_i, 1 \leq i \leq n, |V_i| \leq 2$, since V_i can contain u_i (or) u but not both.

Therefore $\sum_{i=1}^{n} |V_i| = |\nabla(G)| = 2n + 1$, which is a contradiction. Therefore $\chi(G) = n + 1$.

Let $C_1 = \left\{\{v_1, u_1\}, \dots, \{v_n, u_n\}, \{u\}\right\}$.

$C_i = \left\{\{v_i, u\}, \{v_1, u_1\}, \dots, \{v_{i-1}, u_{i-1}\}, \{v_{i+1}, u_{i+1}\}, \dots, \{v_n, u_n\}, \{u_i\}\right\}$.

$C_i = \left\{\{u_1, u_2, \dots, u_n\}, \{v_1, u\}, \{v_2\}, \dots, \{v_i\}, \dots, \{v_n\}\right\}$.

From these we see that $K_{1,n}$ is an induced subgraph of G.

3.24. Theorem

$G = P_n \cup K_1$ can be embedded in a χ-excellent graph.

Proof.

Let us add s vertices to $P_n \cup K_1$ such that the total number of vertices $s + n + 1$ is odd and we get the cycle. Since we know that the any odd cycle is χ-excellent. Hence we get the result.

3.25. Theorem

If G is χ-excellent then $\mu^h(G)$ is χ-excellent.

Proof.

Let us take $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(\mu^h(G)) = \{u_1, u_2, \dots, u_n, u'_1, u'_2, \dots, u'_n, v\}$.

Let G is χ-excellent, and $\chi(G) = k$ then $C = \left\{\{u_i\}, V_2, V_3, \dots, V_k\right\}$ be a χ-partition of G. Then clearly $\chi(\mu^h(G)) = \chi(G) + 1$ and hence $\chi(\mu^h(G)) = k + 1$. The following partitions are the χ-partitions of $\mu^h(G)$,

$C_i = \left\{\{u_i\}, V_2 \cup \{v\}, V_3, V_4, \dots, V_k, \{u'_1, u'_2, \dots, u'_n\}\right\}$.

$C'_i = \left\{\{u'_i\}, V_2 \cup V'_2, \dots, V_k \cup V'_k, \{u_i, v\}\right\}$.

$C_v = \left\{\{v\}, \{u_i, u'_i\}, V_2 \cup V'_2, \dots, V_k \cup V'_k\right\}$.

Therefore every vertex in $\mu^h(G)$ is appears in a singleton in above χ-partitions of $\mu^h(G)$, hence $\mu^h(G)$ is χ-excellent.

3.26. Theorem

Let G be a simple hesitancy fuzzy graph. G is χ-excellent if and only if G is critical.

Proof.

Let us assume that G be a critical graph with chromatic number χ. Let $u \in V(G)$, then $\chi(G-u) > \chi(G)$. Suppose let $\chi(G-u) = \chi(G) - k, (k \geq 1)$. Let $\{V_1, V_2, \dots, V_{\chi(G)-k}\}$ be a χ-partition of $G-u$, then $\{\{u\}, V_1, V_2, \dots, V_{\chi(G)-k}\}$ be a proper color partition of G. Therefore $\chi(G) \leq \chi(G) - k + 1$. Hence $k \leq 1$ which implies that $k = 1$. Therefore $\{\{u\}, V_1, V_2, \dots, V_{\chi(G)-1}\}$ is a χ-partition of G. Hence G is χ-excellent.

Conversely, assume that G is χ-excellent. Then for any $u \in V(G)$, u is either fixed or free and the end vertices of any edge in the graph are free. We know that for a vertex $v \in G$, v is critical if and only if v is either fixed or free, then $\chi(G-u) < \chi(G)$ for every $v \in V(G)$. Also know that for an edge $e = uv$ such that $\mu_2(uv) \leq \min[\mu_1(u), \mu_1(v)]$, $\gamma_2(uv) \leq \max[\gamma_1(u), \gamma_1(v)]$, $\beta_2(uv) \leq \min[\beta_1(u), \beta_1(v)]$, e is critical if and only if each of u and v is either fixed or free, then $\chi(G-e) < \chi(G)$ for every $e \in E(G)$. Therefore any proper subgraph $H(G)$, $\chi(H) < \chi(G)$. Hence G is critical.

3.27. Theorem

Let G be a hesitancy fuzzy vertex transitive graph with a χ-partition containing a singleton. Then G is χ-excellent.

Proof.

Let C be a partition of G containing say $\{u\}$ where $u \in V(G)$. Let $C = \{\{u\}, S_2, \dots, S_\chi\}$. Let $v \in V(G), v \neq u$. Since G is vertex transitive there exists an automorphism φ such that $\varphi(u) = v$. Let $C = \{\{\varphi(u)\}, \varphi(S_2), \dots, \varphi(S_\chi)\}$. Since φ is an automorphism, $\varphi(S_2), \dots, \varphi(S_\chi)$ are all independent. Therefore there exists a χ-partition containing $\{v\}$. Hence the result.

3.28. Observation

There exists a hesitancy fuzzy vertex transitive which is complete in which there exists a χ-partition containing singleton.

3.29. Observation

There exists a hesitancy fuzzy vertex transitive graph which is not complete in which there exists no χ-partition containing singleton.

3.30. Definition

Let $u, v \in V(G)$. u and v are said to be relatively hesitancy fuzzy fixed if u and v belong to the same set in every χ-partition of G. Relatively hesitancy fuzzy free if u and v belong to the same set in some χ-partition of G, (i.e.) u and v do not belong to the same set in some other χ-partition. Relatively hesitancy fuzzy totally free if u and v do not belong to the same set in any χ-partition.

3.31. Remark

Let G be a χ-excellent graph and $e \in E(G)$. Then $G+e, G-e$ need not be χ-excellent.

3.32. Theorem

Let G be a χ-excellent and let $e = uv \in E(G)$. Suppose $\chi(G-e) = \chi(G)$. Then $G-e$ is χ-excellent.

Proof.

Let G be a χ-excellent graph. Suppose let $C = \{\{u\}, V_2, \dots, V_k\}$ be a χ-partition of G, where $k = \chi$. Let $v \in V_i, 2 \leq i \leq k$. If $|V_i| = 1$ then $C_1 = \{\{u, v\}, V_2, \dots, V_{i-1}, V_{i+1}, \dots, V_k\}$ is a proper color partition of G and hence $\chi(G-e) \leq k - 1 = \chi(G) = k$, a contradiction. Therefore $|V_i| \geq 2$.

Therefore $\{u\}$ belongs to a χ-partition of $G-e$. Similarly it can be proved that $\{v\}$ belongs to a χ-partition of $G-e$. The remaining vertices are χ-good in G and hence in $G-e$. Therefore $G-e$ is χ-excellent.

3.33. Observation

[1]. If $G+H$ is χ-excellent if and only if G and H are χ-excellent.

[2]. If $G+H$ is χ-excellent, then $G+H$ need not be complete.

3.34. Theorem

Given a positive integer $k \geq 2$, there exists a hesitancy fuzzy graph G with $\chi = k$ which is not χ-excellent.

Proof.

Let us consider a complete k-partite hesitancy fuzzy graph G with each partition containing at least two vertices. Then $\chi(G) = k$ which is not χ-excellent.

3.35. Theorem

Given a positive integer $t, l, k \geq 2$, and $k-1$ divides $k+l$, there exists a graph G with $\chi(G) = k, \delta(G) - \chi(G) = l$ and G is not χ-excellent.

Proof.

Let us consider a complete k-partite hesitancy fuzzy graph G with each partition containing exactly $\frac{k+l}{k-1}$ vertices. Then $\delta(G) = k + l$ and $\chi(G) = k$ but which is not χ-excellent.

3.36. Observation

A uniquely colorable graph G is χ-excellent if and only if $G = K_n$.

3.37. Observation

If G is χ-excellent then $G \times K_2$ need not be χ-excellent.

3.38. Theorem

Suppose G is χ-excellent and $|V(G) \geq 2|$, then G^+ is not χ-excellent.

Proof.

By the definition of G^+, clearly $\chi(G^+) = \chi(G)$. Since G is χ-excellent, let $C = \left\{\{u\}, S_2, S_3, \ldots, S_\chi\right\}$ where $u \in V(G)$, be a χ-partition of G. Let u' be a pendent vertex adjacent to u. Suppose $C_1 = \left\{\{u'\}, T_2, \ldots, T_\chi\right\}$ be χ-partition of G^+. Then $C_2 = \left\{T_2 \cap V(G), T_3 \cap V(G), \ldots, T_\chi \cap V(G)\right\}$ is a χ-partition of G which is a contradiction. Hence G^+ is not χ-excellent.

3.39. Remark

The only χ-good vertices of G^+ are those of G.

4. CONCLUSION

Thus in this paper we have introduced and analyzed the new concept of chromatic excellence in hesitancy fuzzy graphs by taking into account of hesitancy fuzzy chromatic partition. Hesitancy fuzzy chromatic excellence has been tested in hesitancy fuzzy corona, Mycielskian hesitancy fuzzy graph, union of two hesitancy fuzzy graphs and addition of two hesitancy fuzzy graphs. Since this paper paves way for further studies in chromatic excellence under new parameters.

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