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Transformation Properties of Aging Classes under Excess Wealth Transform

U. Rizwan[1] ., B.Towfeeq Ahmed[2] ., Syed Tahir Hussainy[3],
[1,2,3] *Department of Mathematics, Islamiah College, Vaniyambadi, India.*
[1] *e-mail: towfeesharik@gmail.com*

Abstract:- In this paper, we transform the properties of some aging classes of life distributions into the corresponding properties of excess wealth transform.

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1. INTRODUCTION

Stochastic models are usually sufficiently complex in various fields of statistics, particularly in reliability theory. Obtaining bounds and approximations of their characteristics is of practical importance. That is, the approximation of a stochastic model either by a simpler model or by a model with simple constituent components might lead to convenient bounds and approximations for some particular and desired characteristics of the model.

The study of changes in the properties of a model is also of great interest. Accordingly, since the stochastic components of models involve random variables, the topic of stochastic orders among random variables play an important role in these areas. (see, Muller and Stoyan [1996] and Shaked and Shanthikumar [2007] for an exhaustive monograph on this topic).

Two such well-known stochastic orders are the Total Time on Test transform order and Excess Wealth order, also known in literature as right spread order, and the Total Time on Test transform order whose definitions are recalled here. Throughout this paper, X and Y are two random variables having life distributions F and G, respectively, and denote by $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, their respective survival functions and by $F^{-1}(\cdot)$ and $G^{-1}(\cdot)$ their corresponding right continuous inverses. Moreover, we will use the term *increasing* in place of *non-decreasing*, and *decreasing* in place of *non-increasing.*

The rest of the paper is organized as follows. In section 2, we present some preliminaries of some ageing classes of life distributions. In section 3, we present the transformation properties of the ageing classes under Excess Wealth Transformation. Finally, conclusion is given in section 4.

2. DEFINITIONS AND SOME RELATED CONCEPTS

In reliability theory, aging is usually characterized by a nonnegative random variable $X \geq 0$ with cumulative distribution function $F(\cdot)$ and survival function $\overline{F}(\cdot) = 1 - F(\cdot)$. For any random variable X, let

$$X_t = [X - t \mid X > t], \qquad t \in \{x: F(x) < 1\}$$

Denote a random variable whose distribution is the same as the conditional distribution of X - t given that $X > t$, when X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time *t*, given that the device has survived up to time *t*. Its survival function is

$$\overline{F}_t(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)} \qquad \overline{F}(t) > 0,$$

where $\overline{F}(x)$ is the survival function of X.

Definition 2.1 The failure rate $r(\cdot)$ of a random variable T with distribution function $F(\cdot)$ is usually defined by

$$r(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} P(t < T \leq t + \Delta / T > t) = \frac{f(t)}{\overline{F}(t)}, \qquad (t \geq 0)$$

where $F(t) < 1$, for all $t \geq 0$ and $f(\cdot)$ denotes its density function.

Definition 2.2 A life distribution $F(\cdot)$ and its survival function $\overline{F}(\cdot) = 1 - F(\cdot)$ with support

$S = \{t : \overline{F}(t) > 0\}$ and finite mean $\mu = \int_0^\infty \overline{F}(t)\, dt$ is said to have

- Increasing Failure Rate of order two (IFR(2)), if

$$\overline{F}(t)\int_0^x \overline{F}(u+s)du \geq \overline{F}(s)\int_0^x \overline{F}(u+t)du \quad x\geq 0, t\geq s$$

- New Better than Used of order three (NBU (3)), if

$$\int_0^\infty \int_0^u \overline{F}(x+t)\ dt\ du \leq \overline{F}(x)\int_0^\infty \int_0^u \overline{F}(t)\ dt\ du \quad t,u\geq 0$$

- New Better than Used Convex (NBUC), if

$$\int_y^\infty \overline{F}\ (x+t)\ dt \leq \overline{F}\ (x)\int_y^\infty \overline{F}\ (t)\, dt, \text{ for all } x,y\geq 0$$

- New Better than Used in the increasing convex average (NBUCA), if

$$\int_0^\infty \int_x^\infty \overline{F}(u+t)\ du\ dx \leq \leq \overline{F}(t)\int_0^\infty \int_x^\infty \overline{F}(u)\ du \quad t\geq 0.$$

- New Better than its Equilibrium Life in Convex order (NBELC) if

$$\int_x^\infty \int_y^\infty \overline{F}(u)\ du\ dy \leq \int_x^\infty \overline{F}(y)\ dy \quad x,y\geq 0.$$

- Renewal New is Better than Renewal Used (RNBRU) if

$$\mu\int_{x+t}^\infty \overline{F}(u)\ du \leq \int_x^\infty \overline{F}(u)\, du\ \int_t^\infty \overline{F}(u)\, du \quad x,t\geq 0.$$

- New Better than Used Mean Residual Life (NBUMRL) if

$$\frac{\int_t^\infty \overline{F}\,(x)\, dx}{\overline{F}\,(t)} \leq \mu \quad \text{for } t\geq 0.$$

- New Better than Average Mean Residual Life (NBAMRL) if

$$t^{-1}\int_0^t \frac{\int_t^\infty \overline{F}\,(x)\, dx}{\overline{F}\,(t)}\ dx \leq \mu \quad \text{for } t\geq 0.$$

- Decreasing Harmonic Mean Residual Life (DHMRL) if

$$t^{-1}\int_0^t [\,\mu(x)]^{-1}\ dx \quad \text{is decreasing in } t$$

- Decreasing Mean Residual Life Average (DMRLA) if

$$t^{-1}\int_0^t \mu(x)\ dx \quad \text{is decreasing in } t$$

- Harmonic New Better than Used in Expectation of order three (HNBUE(3) if

$$\int_u^\infty \int_t^\infty \overline{F}\,(x)\ dx\, dt \leq \mu^2 \exp\left(-\frac{u}{\mu}\right)$$

for $u,t\geq 0$.

- Harmonic New Better than Used in Expectation of order two (HNBUE(2) if

$$\int_0^u \int_t^\infty \overline{F}\,(x)\ dx\ dt \leq \mu^2\ \left(1-\exp\left(-\frac{u}{\mu}\right)\right)$$

for $u,t\geq 0$.

- Harmonic New Better than Used in Convex (HNBUC) if

$$\int_t^\infty \overline{F}(x+t)\, dx\, dt \leq \overline{F}(t)\int_t^\infty \overline{F}(x)\, dx \quad t\geq 0$$

- New Better than Average Renewal Failure Rate (NBARFR) if

$$\int_0^\infty \overline{F}\,(u)\, du \leq \exp(-r(0)\, x) \text{ for } x\geq 0.$$

where $r(0)$ non-zero initial failure rate and μ denotes the finite mean.

- Generalized New Better than Average Renewal Failure Rate (GNBARFR) if

$$\int_0^\infty \int_t^\infty \overline{F}(u)\ du\ dt \leq \frac{\mu}{r(0)} \quad t>0$$

where $r(0)$ non-zero initial failure rate and μ is finite mean.

- New Better than Used Average Renewal Failure Rate in order Three (NBUARFR (3)), if

$$\int_x^\infty \int_u^\infty \overline{F}(u)\ du\ dx \leq \frac{\mu}{r(0)}\exp(-r(0)\, x)$$

$x,u\geq 0$

where $r(0)$ non-zero initial failure rate and μ is finite mean.

- New Better than Used Renewal Failure Rate (NBURFR)

$$\int_y^\infty \overline{F}\,(u)\, du \leq \frac{\overline{F}\,(y)}{r(0)} \quad \text{for } y\geq 0$$

- Expectation Better than Used in Convex order two (EBUC (2)) if

$$\int_u^\infty \overline{F}(x+t)\ dt \le \exp(-x/\mu)\int_u^\infty \overline{F}(t)\ dt \quad \text{for } t,u \ge 0.$$

- Expectation Better than Used Aged order two (EBUA (2)) if

$$\int_0^\infty \int_0^u \overline{F}(x+t)\ dx\ \ dt \le \mu^2\ (1-\exp(-u/\mu))\quad t,u \ge 0$$

- Used Better than Aged in Convex Tail ordering (UBACT) if

$$\int_x^\infty \int_z^u \overline{F}(u+t)\ du\ \ dz \le \gamma^{-2}\,\overline{\mathrm{F}}(\mathrm{t})\ \ \exp(-\gamma\, x)\quad t,z \ge 0$$

where γ is asymptotic decay of X.

- Used Better than Aged Expectation (UBAE) if

$$\int_x^\infty \overline{F}(u)\ du \le \mu\ \exp(-x/\mu) \qquad \text{for all } x \ge 0$$

- Exponential Better than Used in Convex Average (EBUCA), if

$$\int_0^\infty \int_{x+t}^u \overline{F}(u)\ du\ \ dx \le \mu\ \ \overline{\mathrm{F}}(\mathrm{t})\int_0^\infty \exp(-x/\mu)\ dx\quad x,t \ge 0$$

- Harmonic Used Better than aged in Expectation in upper Tail (HUBAEUT)

$$\int_x^\infty \int_t^\infty \overline{F}(u)\,du\,dt \ge \frac{\mu}{\gamma}\exp(-\gamma x)\ ,$$

for all $x,t \ge 0$, where γ is asymptotic decay of X.

- Harmonic New Better than Renewal Used Expectation (HNBRUE), If

$$2\ \mu\ \int_x^\infty \int_y^\infty \overline{F}(u)\,du\ dy \le \mu_{(2)}\exp(-x/\mu)$$

for all $x,y \ge 0$.

- Renewal Used Better than Expectation (RNBUE) if

$$2\ \ \mu \int_x^\infty \overline{F}(u)\,du \le \mu_{(2)}\ \overline{F}(x)\quad x \ge 0$$

where μ is the mean and $\mu_{(2)}$ is second moment, both assumed to be finite.

- Renewal New Better than Renewal Used Expectation (RNBRUE) if

$$2\ \mu \int_0^\infty \int_u^\infty \overline{F}\ (w)dw\ \ du \le \mu_{(2)}\ \int_x^\infty \overline{F}\ (u)\ \ du \quad \text{for } x,u \ge 0$$

where μ is the mean and $\mu_{(2)}$ is second moment, both assumed to be finite.

- New Better than Equilibrium in Expectation (NBEE), if

$$\int_0^\infty \int_t^\infty \overline{F}(u)\ du\ dt \le \mu^2 \quad \text{for } t \ge 0$$

- Harmonic New Better than Renewal Used Expectation in Starshaped (HNBRUES), If

$$y\int_y^\infty \overline{F}\ (u)\ du + \int_y^\infty \int_u^\infty \overline{F}\ (s)\,ds\ du \le \mu\ (y+1)\exp(-y/\mu) \text{ for } u,y \ge 0.$$

- Decreasing Variance Residual Life (DVRL) if

$$\overline{F}\ (t)\int_t^\infty \int_y^\infty \overline{F}\ (x)\,dx\ dy \le \left(\int_t^\infty \overline{F}\ (u)\,du\right)^2 \quad \text{for } t,y \ge 0$$

- New Better than Used At Specific Interval (NBUASI), if

$$\int_y^{x+y} \overline{F}\ (t+u)\,du \le \overline{F}\ (t)\ \int_y^{y+x} \overline{F}\ (u)\,du, \quad \text{for } x,y \ge 0$$

3. EXCESSS WEALTH TRANSFORM

We transform the properties of aging classes of stochastic life time models into the corresponding properties of the Excess Wealth Transform. The Excess Wealth Transform is useful for identification of failure distribution models. Let $F(\cdot)$ be a life distribution with finite mean :

The Excess Wealth Transform H_F^{-1} of $F(\cdot)$ is defined by

$$H_F^{-1}\ (p) = \int_{F^{-1}(p)}^\infty \overline{F}\ (s)\ \ ds$$

for $0 \le p \le 1$,

where $F^{-1}(t) = \inf\{\ x : F(x) \in t\ \}$. Since the mean is given by

$$\mu = H_F^{-1}\ (0) = \int_{F^{-1}(0)}^\infty \overline{F}\ (s)\ \ ds$$

the transform,

$$\psi_F(p) = \frac{H_F^{-1}(p)}{H_F^{-1}(0)}$$

$$= \frac{1}{\mu} \int_{F^{-1}(p)}^{\infty} \bar{F}(s) \, ds \qquad (3.1)$$

is scale invariant and is called scaled Excess Wealth Transform

In this section, we transform the properties of the aging classes. Namely, IFR(2), NBU(3), NBEE, RNBRU, HNBUC, NBELC, NBUCA, NBARFR, NBURFR, NBUMRL, HNBUE(3), DVRL, DHMRLA, NBUHMRL, HNBUES, HNBUE(2), GNBARFR, NBARFR(3), EBUCA(2), EBUA(2), UBACT, EBU(2), UBAE, EBUCA, HUBAEUT,HNBRUE, RNBRUE, NBUC, into the corresponding properties of Excess Wealth Transform.

Remark : From equation (3.1), when $p = 0$, then $\psi_F(0) = 1$ and when $p = 1$, then $\psi_F(1) = 0$

Theorem 3.1 *A life distribution F is* IFR (2) *if and only if*

$$(1-w)\left\{\psi_F(p) - \frac{1}{u}\int_{F^{-1}(p)+F^{-1}(q)}^{\infty} \bar{F}(s)ds\right\} \geq (1-p)\left\{\psi_F(w) - \frac{1}{u}\int_{F^{-1}(q)+F^{-1}(w)}^{\infty} \bar{F}(u)du\right\}$$

for $0 \leq p, q, w \leq 1$.

Proof: Suppose that F is IFR (2). Then,

$$\bar{F}(t)\int_0^x \bar{F}(u+s)\,du \geq \bar{F}(s)\int_0^x \bar{F}(u+t)\,du$$

for $x, t \geq 0$

$$\bar{F}(t)\left\{\int_0^{x+s} \bar{F}(u)\,du - \int_0^s \bar{F}(u)\,du\right\} \geq \bar{F}(s)\left\{\int_0^{x+t} \bar{F}(u)\,du - \int_0^t \bar{F}(u)\,du\right\}$$

$$\bar{F}(t)\left\{\int_s^{\infty} \bar{F}(u)\,du - \int_{x+s}^{\infty} \bar{F}(u)\,du\right\} \geq \bar{F}(s)\left\{\int_t^{\infty} \bar{F}(u)\,du - \int_{x+t}^{\infty} \bar{F}(u)\,du\right\}$$

Let $F(s) = p, F(x) = q$ and $F(t) = w$, where $0 \leq p, q, w \leq 1$. Then

$$(1-w)\left\{\psi_F(p) - \frac{1}{u}\int_{F^{-1}(p)+F^{-1}(q)}^{\infty} \bar{F}(s)ds\right\} \geq (1-p)\left\{\psi_F(w) - \frac{1}{u}\int_{F^{-1}(q)+F^{-1}(w)}^{\infty} \bar{F}(u)du\right\}$$

This completes the proof of the theorem. ■

Theorem 3.2 *A life distribution F is* NBU (3) *if and only if*

$$\int_0^1 \left\{\psi_F(p) - \frac{1}{u}\int_{F^{-1}(p)+F^{-1}(q)}^{\infty} \bar{F}(s)\,ds\right\} \frac{\psi_F'(q)}{(1-q)}\,dq \leq (1-p)\int_0^1 \frac{(1-\psi_F(q))\psi_F'(q)}{(1-q)}\,dq$$

where $0 \leq p, q < 1$.

Proof. Suppose that F is NBU (3) . Then

$$\int_0^{\infty}\int_0^u \bar{F}(x+t)\,dt\,du \leq \bar{F}(x)\int_0^{\infty}\int_0^u \bar{F}(t)\,dt\,du \qquad u, t \geq 0$$

$$\int_0^{\infty}\left\{\int_x^{x+u} \bar{F}(s)\,ds\right\} du \leq \bar{F}(x)\int_0^{\infty}\int_0^u \bar{F}(t)\,dt\,du$$

$$\int_0^{\infty}\left\{\int_x^{\infty} \bar{F}(s)\,ds - \int_{x+u}^{\infty} \bar{F}(s)\,ds\right\} du \leq \bar{F}(x)\int_0^{\infty}\left\{\mu - \int_u^{\infty} \bar{F}(t)\,dt\right\} du$$

Let $F(x) = p$, and $F(u) = q$,
$du = \dfrac{dq}{f[F^{-1}(q)]}$. Then

$$\int_0^1 \left\{\psi_F(p) - \frac{1}{u}\int_{F^{-1}(p)+F^{-1}(q)}^{\infty} \bar{F}(s)ds\right\} \frac{\psi_F'(q)}{(1-q)}\,dq \leq (1-p)\int_0^1 \frac{(1-\psi_F(q))\psi_F'(q)}{(1-q)}\,dq$$

This completes the proof of the theorem. ■

Theorem 3.3 *A life distribution F is* NBEE *if and only if*

$$\int_0^1 \left(\frac{\psi_F(p)\, \psi'_F(p)}{(1-p)} \right) dp \geq -1,$$

for $0 \leq p < 1$.

Proof. Suppose that F is NBEE. Then

$$\int_0^\infty \int_t^\infty \bar{F}(x)\, dx\, dt \leq \mu^2 \qquad \text{for all } t \geq 0$$

Let $F(t) = p, where\ 0 \leq p < 1$.

Then $dt = \frac{dp}{f[F^{-1}(p)]}$, and

$$\int_0^1 \int_{F^{-1}(p)}^\infty \bar{F}(x)\, dx \frac{dp}{f[F^{-1}(p)]} \leq \mu^2$$

$$\int_0^1 \mu\, \psi_F(p) \frac{dp}{f[F^{-1}(p)]} \leq \mu^2.$$

$since\ [f[F^{-1}(p)]]^{-1} = -\frac{\mu\, \psi_F{}'(p)}{(1-p)}$, we obtain on simplification that

$$\int_0^1 \left(\frac{\psi_F(p)\, \psi'_F(p)}{(1-p)} \right) dp \geq -1.$$

This completes the proof of the theorem. ■

Theorem 3.4 *A life distribution F is* RNBRU *if and only if*

$$\left\{ \frac{1}{\mu} \int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(u) du \right\} \leq \psi_F(p)\ \psi_F(q)$$

for $0 \leq p,\ q \leq 1$.

Proof. Suppose that F is RNBRU. Then,

$$\mu \int_{x+t}^\infty \bar{F}(u)\, du$$

$$\leq \int_x^\infty \bar{F}(u)\, du \int_t^\infty \bar{F}(u)\, du, \text{ for } x, t \geq 0.$$

On dividing by μ which is positive and finite, and let $F(x) = p, F(t) = q$, where $0 \leq p, q \leq 1$, we have

$$\left\{ \frac{1}{\mu} \int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(u) du \right\} \leq \psi_F(p)\ \psi_F(q)$$

This completes the proof of the theorem. ■

Theorem 3.5 *A life distribution F is* HNBUC *if and only if*

$$\left\{ \frac{1}{\mu} \int_{2\ F^{-1}(p)}^\infty \bar{F}(x)\, dx \right\} \leq (1-p)\ \psi_F(p)$$

for $0 \leq p \leq 1$.

Proof. Suppose that F is HNBUC. Then,

$$\int_t^\infty \bar{F}(x+t)\, dx \leq \bar{F}(t) \int_t^\infty \bar{F}(x)\, dx, \text{ for all } t \geq 0.$$

$$\left\{ \int_{2t}^\infty \bar{F}(x)\, dx \right\} \leq \bar{F}(t) \left\{ \int_t^\infty \bar{F}(x)\, dx \right\}$$

Let $F(t) = p$, where $0 \leq p \leq 1$. Then

$$\left\{ \frac{1}{\mu} \int_{2\ F^{-1}(p)}^\infty \bar{F}(x)\, dx \right\} \leq (1-p)\ \psi_F(p)$$

This completes the proof of the theorem. ■

Theorem 3.6 *A life distribution F is* NBELC *if and only if*

$$\int_q^1 \left(\frac{\psi_F(p)\, \psi'_F(p)}{(1-p)} \right) dp \geq -\ \psi_F(q)$$

for $0 \leq p, q < 1$.

Proof. Suppose that F is NBELC. Then

$$\int_x^\infty \int_y^\infty \bar{F}(u)\, du\, dy \leq \mu \int_x^\infty \bar{F}(y) dy \quad \text{for } x, y \geq 0$$

Let $F(y) = p$ and $F(x) = q, dy = \frac{dp}{f[F^{-1}(p)]}$, where $0 \leq p \leq 1$. Then

$$\int_{F(x)}^1 \{\mu \psi_F(p)\} \frac{dp}{f[F^{-1}(p)]} \leq \mu \quad (\mu \quad \psi_F(q))$$

Since $[f[F^{-1}(p)]]^{-1} = -\frac{\mu\ \psi'_F(p)}{(1-p)}$,

we obtain on simplification that

$$\int_q^1 \left(\frac{\psi_F(p)\, \psi'_F(p)}{(1-p)} \right) dp \geq -\ \psi_F(q)$$

This completes the proof of the theorem. ■

Theorem 3.7 *A life distribution F is* NBUCA *if and only if*

$$\int_0^1 \left\{ 1 - \frac{1}{u} \int_0^{F^{-1}(p)+F^{-1}(q)} \bar{F}(u) du \right\} \frac{\varphi'_F(p)}{(1-p)}\ dp$$

$$\geq (1-q)\int_0^1 \{1-\varphi_F(p)\}\frac{\varphi_F'(p)}{(1-p)}\,dp$$

for $0 \leq p, q < 1$.

Proof. Suppose that F is NBUCA. Then,

$$\int_0^\infty \int_x^\infty \bar{F}(u+t)\,du\,dx \leq \bar{F}(t)\int_0^\infty \int_x^\infty \bar{F}(u)\,du\,dx \qquad \text{for} \quad x, t \geq 0$$

$$\int_0^\infty \left\{\int_{x+t}^\infty \bar{F}(u)\,du\right\} dx \leq \bar{F}(t)\int_0^\infty \left\{\int_x^\infty \bar{F}(u)\,du\right\} dx$$

Let $F(x) = p$ and $F(t) = q$, then $dx = \frac{dp}{f[F^{-1}(p)]}$

$$\int_0^1 \left\{\int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(u)du\right\}\frac{dp}{f[F^{-1}(p)]} \leq (1-q)\int_0^1 \mu\,\psi_F(p)\,\frac{dp}{f[F^{-1}(p)]}$$

Since $[f[F^{-1}(p)]]^{-1} = -\frac{\mu\ \psi_F'(p)}{(1-p)}$, we obtain on simplification that

$$\int_0^1 \left\{\int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(u)du\right\}\frac{\psi_F'(p)}{(1-p)}\,dp \geq (1-q)\int_0^1 \psi_F(p)\,\frac{\psi_F'(p)}{(1-p)}dp$$

This completes the proof of the theorem. ■

Theorem 3.8 *A life distribution F is* NBARFR *if and only if*

$$\psi_F(p) \leq \ exp\,(-r(0)\,F^{-1}(p))$$

for $0 \leq p \leq 1$, where $r(0)$ is the non-zero initial failure rate and μ denotes the finite mean.

Proof. Suppose that F is NBARFR. Then

$$\int_0^\infty \int_x^\infty \bar{F}(u)\,du\,dt \leq \mu\ \exp(-r(0)\cdot x)\,,$$

for all $t \geq 0$

Let $F(x) = p,$ where $0 \leq p \leq 1$. Then

$$(\psi_F(p)) \leq \ exp(-r(0)\,F^{-1}(p))$$

This completes the proof of the theorem. ■

Theorem 3.9 *A life distribution F is* NBURFR *if and only if*

$$\psi_F(p) \leq (1-p)/\mu\ r(0)$$

for $0 \leq p \leq 1$, where $r(0)$ is non-zero initial failure rateof F and μ denotes the finite mean.

Proof. Suppose that F is NBURFR. Then

$$\int_y^\infty \bar{F}(u)\,du \leq \bar{F}(y)\,/\,r(0)$$

Let $F(y) = p,$ where $0 \leq p \leq 1$. Then

$$\psi_F(p) \leq (1-p)/\mu\ r(0)$$

This completes the proof of the theorem. ■

Theorem 3.10 *A life distribution F is* NBUMRL *if and only if*

$$\psi_F(p) \leq (1-p) \quad \text{for } 0 \leq p \leq 1.$$

Proof. Suppose that F is NBUMRL. Then

$$\frac{\int_t^\infty \bar{F}(x)\ dx}{\bar{F}(t)} \leq \mu \quad \text{for} \quad t \geq 0$$

Let $F(t) = p,$ where $0 \leq p \leq 1$. Then

$$\psi_F(p) \leq (1-p)$$

This completes the proof of the theorem. ■

Theorem 3.11 *A life distribution F is* HNBUE(3) *if and only if*

$$\int_{F(u)}^1 \frac{\psi_F(p)\cdot\psi_F'(p)}{(1-p)}\,dp \geq -\ \exp(-u/\mu)$$

for $0 \leq p < 1$, where μ denotes the finite mean.

Proof. Suppose that F is HNBUE(3) . Then

$$\int_u^\infty \int_t^\infty \bar{F}(x)\,dx\ dt \leq \mu^2\exp(-u/\mu)$$

for $u, t \geq 0$.

Let $F(t) = p,$ where $0 \leq p \leq 1$. Then

$$\int_{F(u)}^1 \int_{F^{-1}(p)}^\infty \bar{F}(x)\,dx\frac{dp}{f[F^{-1}(p)]} \leq \mu^2\exp(-u/\mu)$$

Since $[f[F^{-1}(p)]]^{-1} = -\frac{\mu\ \psi_F'(p)}{(1-p)}$, we have

$$\int_{F(u)}^1 \frac{\psi_F(p)\cdot\psi_F'(p)}{(1-p)}\,dp \geq -\exp(-u/\mu)$$

This completes the proof of the theorem. ■

The proofs of theorems 3.12 to 3.17 are on similar lines as above and hence are omitted.

Theorem 3.12 *A life distribution F is* DVRL *if and only if*

$$\frac{(\psi_F\ (q)\ )^2}{(1-q)} \leq -\int_q^1 \frac{\psi_F\ (p)\cdot\psi_F'(p)}{(1-p)}\ dp$$

for $0 \leq p, q < 1$,

Proof. Suppose that F is DVRL . Then

$$\overline{F}\ (t) \int_t^\infty \int_y^\infty \overline{F}\ (x)\ dx\ dy \leq \left(\int_t^\infty \overline{F}\ (u)\ du \right)^2 \text{ for } t, y \geq 0$$

Let $F(y) = p, F(t) = q$, where $0 \leq p, q < 1$, $dy = \frac{dp}{f[F^{-1}(p)]}$. Then

$$(1-q) \int_{F(t)}^1 \int_{F^{-1}(p)}^\infty \overline{F}\ (x)\ dx\ \frac{dp}{f[F^{-1}(p)]} \leq \left(\int_{F^{-1}(q)}^\infty \overline{F}\ (u)\ du \right)^2$$

since $\left[f[F^{-1}(p)]\right]^{-1} = -\frac{\mu\ \psi_F'\ (p)}{(1-p)}$,

we obtain on simplification that

$$\frac{(\psi_F\ (q)\ )^2}{(1-q)} \leq -\int_q^1 \frac{\psi_F\ (p)\cdot\psi_F'(p)}{(1-p)}\ dp$$

This completes the proof of the theorem. ■

Theorem 3.13 *A life distribution F is* DHMRLA *if and only if*

$$\frac{\ln(\psi_F\ (F(t))}{t} \text{ is decreasing in } t$$

Theorem 3.14 *A life distribution F is* DMRLA *if and only if*

$$\mu^2\ t^{-1} \int_0^{F(t)} \frac{\psi_F\ (p)\cdot\psi_F'(p)}{(1-p)^2}\ dp \text{ is increasing in } t$$

Theorem 3.15 *A life distribution F is* NBUHMRL *if and only if*

$$(\psi_F\ (F(t)\ )^{-1} \leq \exp(t/\mu) \text{ for } 0 < t < \infty.$$

Theorem 3.16 *A life distribution F is* NBUASI *if and only if*

$$\psi_F\ (q) \geq \left(\mu\ (1-p)\right)^{-1} \left\{ \int_{F^{-1}(p)+F^{-1}(w_1)}^\infty \overline{F}\ (s)\ ds - \int_{F^{-1}(p)+F^{-1}(q)+F^{-1}(w_1)}^\infty \overline{F}\ (s)\ ds \right\} + \frac{1}{\mu}\int_{F^{-1}(w_2)+F^{-1}(q)}^\infty \overline{F}\ (u)\ du$$

for $0 \leq p, q, w_1, w_2 < 1$.

Theorem 3.17 *A life distribution F is* HNBUES *if and only if*

$$\psi_F\ (p) \geq (F^{-1}(p))^{-1} \left\{ \mu \int_p^1 \frac{\psi_F(q)\cdot\psi_F'(q)}{(1-q)}\ dq + (1+F^{-1}(p) \cdot exp(-F^{-1}(p)/\mu) \right\}$$

for $0 \leq p, q < 1$.

Theorem 3.18 *A life distribution F is* HNBUE(2) *if and only if*

$$\int_0^{F(u)} \frac{\psi_F\ (p)\cdot\psi_F'(p)}{(1-p)}\ dp \geq (\ \exp(-u\ /\mu) - 1)$$

for $0 \leq p < 1$, where μ denotes finite mean of F.

Proof. Suppose that F is HNBUE(2) . Then

$$\int_u^\infty \int_t^\infty \overline{F}\ (x)\ dx\ dt \leq \mu^2\ (1 - \exp(-u\ /\ \mu)\ ) \text{ for } u, t \geq 0$$

Let $F(t) = p$, where $0 \leq p \leq 1$. Then

$$\int_0^{F(u)} \int_{F^{-1}\ (p)}^\infty \overline{F}\ (x)\ dx \frac{dp}{f[F^{-1}(p)]} \leq \mu^2\ (1 - \exp(-u\ /\ \mu)\ )$$

since $\left[f[F^{-1}(p)]\right]^{-1} = -\frac{\mu\ \psi_F'\ (p)}{(1-p)}$,

we obtain on simplification that

$$\int_0^{F(u)} \frac{\psi_F\ (p)\cdot\psi_F'(p)}{(1-p)}\ dp \geq (\ \exp(-u\ /\mu) - 1)$$

This completes the proof of the theorem. ■

Theorem 3.19 *A life distribution F is* GNBARFR *if and only if*

$$\int_0^1 \psi_F(p)\left(\frac{\psi_F'(p)}{(1-p)}\right) dp \geq -\frac{1}{\mu\, r(0)}$$

for $0 \leq p \leq 1$, where $r(0)$ is the non-zero initial failure rate and μ denotes the finite mean.

Proof. Suppose that F is GNBARFR. Then

$$\int_0^\infty \int_t^\infty \bar{F}(u)\, du\, dt \leq \frac{\mu}{r(0)}, \text{ for all } t > 0$$

Let $F(t) = p$, where $0 \leq p \leq 1$.

Then $dt = \frac{dp}{f[F^{-1}(p)]}$.

Since $[f[F^{-1}(p)]]^{-1} = \left(-\frac{\mu\,\psi_F'(p)}{(1-p)}\right)$, we have

$$\int_0^1 (\mu\ \psi_F(p))\left(-\frac{\mu\ \psi_F'(p)}{(1-p)}\right) dp \leq \frac{\mu}{r(0)}$$

$$\int_0^1 \psi_F(p)\left(\frac{\psi_F'(p)}{(1-p)}\right) dp \geq -\frac{1}{\mu\, r(0)}$$

This completes the proof of the theorem. ■

Theorem 3.20 *A life distribution F is* NBARFR (3) *if and only if*

$$\int_{F(y)}^1 \frac{\psi_F(p)\cdot \psi_F'(p)}{(1-p)}\, dp \geq -\frac{1}{\mu\ \ r(0)} \exp(-r(0)\,.\,y)$$

for $0 \leq p < 1$, where $r(0)$ is the non-zero initial failure rate and μ denotes the finite mean.

Proof. The proof is similar and hence is omitted. ■

Theorem 3.21 *A life distribution F is* EBUC (2) *if and only if*

$$\psi_F(q) \geq \exp(F^{-1}(p)/\mu)\left(\frac{1}{\mu}\int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(s)\ ds\right)$$

for $0 \leq p, q \leq 1$.

Proof. Suppose that F is EBUC (2). Then,

$$\int_{x+u}^\infty \bar{F}(s)ds \leq \exp(-x/\mu)\int_u^\infty \bar{F}(t)dt$$

$$for\ x, u \geq 0$$

Let $F(x) = p$, and $F(u) = q$, where $0 \leq p, q \leq 1$. Then

$$\int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(s)ds \leq \exp(-F^{-1}(p)/\mu)\int_{F^{-1}(q)}^\infty \bar{F}(t)dt$$

$$\psi_F(q) \geq \exp(F^{-1}(p)/\mu)\left(\frac{1}{\mu}\int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(s)\ ds\right)$$

This completes the proof of the theorem. ■

Theorem 3.22 *A life distribution F is* EBUA (2) *if and only if*

$$\int_0^1 \left\{\psi_F(p) - \frac{1}{\mu}\int_{F^{-1}(p)+F^{-1}(q)}^\infty \bar{F}(s)ds\right\}\frac{\psi_F'(q)}{(1-q)}\, dq \geq (1-\exp(-F^{-1}(p)/\mu))$$

for $0 \leq p, q \leq 1$.

Proof. The proof is on similar lines to that of the above Theorem and is omitted. ■

Theorem 3.23 *A life distribution F is* UBACT *if and only if*

$$\int_{F(x)}^1 \left\{\frac{1}{\mu}\int_0^{F^{-1}(p)+F^{-1}(q)} \bar{F}(s)ds\right\}\frac{\psi_F'(p)}{(1-p)}\, dp \leq \frac{\gamma^{-2}}{\mu}\ (1-q)\exp(-\gamma\, x).$$

Where γ is asymptotic decay of X, for $0 \leq p, q < 1$.

Proof. Suppose that F is UBACT. Then,

$$\int_x^\infty \int_{z+t}^\infty \bar{F}(s)\, ds \geq \gamma^2\, \bar{F}(\mathrm{t})\exp(-\gamma x)$$

for all $x, t, z \geq 0$

Let $F(z) = p$ and $F(t) = q$.

Then $dz = \frac{dp}{f[F^{-1}(p)]}$ where $0 \leq p, q < 1$.

Since $[f[F^{-1}(p)]]^{-1} = \frac{\mu\,\varphi_F'(p)}{(1-p)}$,

on simplification we obtain

$$\int_{F(x)}^{1}\left\{\frac{1}{\mu}\int_{0}^{F^{-1}(p)+F^{-1}(q)}\bar{F}(s)ds\right\}\frac{\psi_F'(p)}{(1-p)}\,dp \le \frac{\gamma^{-2}}{\mu}\ (1-q)\exp(-\gamma\, x)$$

This completes the proof of the theorem. ■

Theorem 3.24 *A life distribution F is* EBU (2) *if and only if*

$$\psi_F(p)\ -\ \left(1-\ \psi_F(p)\right)\mu\ \exp(-F^{-1}(p)/\mu\ ) \le \int_{F^{-1}(p)+\,F^{-1}(q)}^{\infty}\overline{F}\,(t)\,dt$$

for $0 \le p,q \le 1$, where μ denotes finite mean.

Proof. Suppose that F is EBU (2). Then

$$\int_{x}^{x+u}\bar{F}(t)dt \le\ \bar{F}(x)\int_{0}^{u}\bar{F}(t)dt \qquad x,u \ge 0$$

Let $F(x) = p$ and $F(u) = q\ , where\ 0 \le p,q \le 1.$ Then

$$\psi_F(p)\ -\ \left(1-\ \psi_F(p)\right)\mu\ \exp(-F^{-1}(p)/\mu\ ) \le \int_{F^{-1}(p)+\,F^{-1}(q)}^{\infty}\overline{F}\,(t)\,dt$$

This completes the proof of the theorem. ■

Theorem 3.25 *A life distribution F is* UBAE *if and only if*

$$\psi_F(p) \ge\ \exp(-F^{-1}(p)/\mu\,) \qquad \text{for} \qquad 0 \le p \le 1.$$

Proof. Suppose that F is UBAE. Then,

$$\int_{x}^{\infty}\bar{F}(x)\ dx \ge \mu\exp(-x/\mu)\ \ \text{for } x \ge 0$$

Let $F(x) = p,$ where $0 \le p \le 1.$ Then

$$\psi_F(p) \ge\ \exp(-F^{-1}(p)/\mu\,)$$

This completes the proof of the theorem. ■

Theorem 3.26 *A life distribution F is* EBUCA *if and only if*

$$\int_{0}^{1}\left\{\frac{1}{\mu}\int_{F^{-1}(p)+F^{-1}(q)}^{\infty}\bar{F}(u)du\right\}\frac{\psi_F'(p)}{(1-p)}\,dp \ge\ (1-q)\ \ \text{for}\ \ 0 \le p,q < 1$$

Proof. Suppose that F is EBUCA. Then

$$\int_{0}^{\infty}\int_{x+t}^{\infty}\bar{F}(u)\,du \ge\ \mu^2\,\bar{F}(t)\ \ ,\ \ \text{for all } x,t \ge 0$$

Let $F(x) = p$ and $F(t) = q$.

$$\text{Then } dx = \frac{dp}{f[F^{-1}(p)]} \text{ where } 0 \le p,q \le 1.$$

Since $\left[f[F^{-1}(p)]\right]^{-1} = \left(-\ \frac{\mu\,\psi_F'(p)}{(1-p)}\right),$

we obtain on simplification that

$$\int_{0}^{1}\left\{\frac{1}{\mu}\int_{F^{-1}(p)+F^{-1}(q)}^{\infty}\bar{F}(u)du\right\}\frac{\psi_F'(p)}{(1-p)}\,dp\ \ \ge\ (1-q).$$

This completes the proof of the theorem. ■

Theorem 3.27 *A life distribution F is* HUBAEUT *if and only if*

$$\mu\ \int_{F(x)}^{1}\psi_F\ (p)\ \frac{\psi_F'(p)}{(1-p)}\ \ dp\ \le\ -\ \frac{1}{\gamma}\ \exp(-\gamma x)$$

for $0 \le p < 1$.

Proof. Suppose that F is HUBAEUT. Then

$$\int_{x}^{\infty}\int_{t}^{\infty}\bar{F}(u)\,du\,dt \ge\ \frac{\mu}{\gamma}\exp(-\gamma x)\quad ,\ \ \text{for all } x,t \ge 0$$

Let $F(t) = p\ , where\ 0 \le p < 1.$ Then $dt = \frac{dp}{f[F^{-1}(p)]}$.

Since $\left[f[F^{-1}(p)]\right]^{-1} = \left(-\ \frac{\mu\,\psi_F'(p)}{(1-p)}\right)$, we have

$$\mu\int_{F(x)}^{1}\psi_F\ (p)\ \frac{\psi_F'(p)}{(1-p)}\ \ dp\ \le\ -\ \frac{1}{\gamma}\ \exp(-\gamma x).$$

This completes the proof of the theorem. ■

Theorem 3.28 *A life distribution F is* HNBRUE *if and only if*

$$2\mu^2\int_{F(x)}^{1}\psi_F\ (p)\ \frac{\psi_F'(p)}{(1-p)}\ \ dp\ \ge\ -\ \mu_{(2)}\ \exp(-x/\mu)$$

for $0 \le p < 1$.

Proof. Suppose that F is HNBRUE. Then

$$2\int_{x}^{\infty}\int_{y}^{\infty}\bar{F}(u)\,du\,dy \le\ \ \mu_{(2)}\exp(-x/\mu),$$

for all $x,t \ge 0$

Let $F(y) = p\ , where\ \ 0 \le p < 1.$

$$\text{Then}\quad dy = \frac{dp}{f[F^{-1}(p)]}.$$

$since\ \left[f[F^{-1}(p)]\right]^{-1} = \left(-\frac{\mu\,\psi'_F(p)}{(1-p)}\right)$, on simplification that

$$2\mu^2 \int_{F(x)}^{1} \psi_F\,(p)\,\frac{\psi'_F(p)}{(1-p)}\;dp \;\geq\; -\,\mu_{(2)}\;\exp(-x/\mu)$$

This completes the proof of the theorem. ■

Theorem 3.29 *A life distribution F is* RNBRUE *if and only if*

$$-\,\psi_F(q) \;\leq\; \frac{2\;\mu^2}{\mu_{(2)}} \int_{F(x)}^{1} \frac{\psi_F\,(p)\;\cdot\;\psi'_F\,(p)}{(1-p)}\;dp,$$

for $0 \leq p, q < 1$, where μ is the mean and $\mu_{(2)}$ is the second moment, both assumed finite.

Proof. Suppose that F is RNBRUE. Then,

$$2\mu \int_{x}^{\infty}\int_{u}^{\infty} \bar{F}(w)\,dw\,du \leq\; \mu_{(2)} \int_{x}^{\infty} \bar{F}(u)\,du\,,$$

for all $x, u \geq 0$

Let $F(u) = p$ and $F(x) = q$, where $0 \leq p, q \leq 1$.

Then $du = \dfrac{dp}{f[F^{-1}(p)]}$.

$since\ \left[f[F^{-1}(p)]\right]^{-1} = \left(-\frac{\mu\,\psi'_F(p)}{(1-p)}\right)$, we have

$$-\,\psi_F(q) \;\leq\; \frac{2\;\mu^2}{\mu_{(2)}} \int_{F(x)}^{1} \frac{\psi_F\,(p)\cdot\psi'_F\,(p)}{(1-p)}\;dp$$

This completes the proof of the theorem. ■

Theorem 3.30 *A life distribution F is* NBUC *if and only if*

$$\frac{1}{\mu\,(1-q)} \int_{F^{-1}(p)+F^{-1}(q)}^{\infty} \overline{F}\;(t)\;\;dt \;\leq\;\; \psi_F\,(q)$$

for $0 \leq p, q < 1$.

Proof. Suppose that F is NBUC. Then

$$\int_{y}^{\infty} \overline{F}\;(x+t)\;dt$$

$$\leq\; \overline{F}\;(x) \int_{y}^{\infty} \overline{F}\,(t)\,dt, \qquad \text{for all}\quad x, y \geq\; 0$$

$$\int_{y+x}^{\infty} \overline{F}\;(t)\;\;dt \;\leq\; \overline{F}\;\;(x) \int_{y}^{\infty} \overline{F}\,(t)\,dt,$$

Let $F(y) = p, F(x) = q$, where $0 \leq p, q < 1$. Then

$$\int_{F^{-1}(p)+F^{-1}(q)}^{\infty} \overline{F}\;(t)\;\;dt$$

$$\leq (1-q) \int_{F^{-1}(q)}^{\infty} \overline{F}\,(t)\,dt,$$

$$\frac{1}{\mu(1-q)} \int_{F^{-1}(p)+F^{-1}(q)}^{\infty} \overline{F}\;\,(t)\;\;dt \;\leq\;\; \psi_F\,(q)$$

This completes the proof of the theorem. ■

4. CONCLUSION

In this paper, we have transformed the properties of certain ageing classes into the corresponding properties of their respective Excess Wealth transforms viz. IFR(2), NBU(3), NBEE, RNBRU, HNBUC, NBELC, NBUCA, NBARFR, NBURFR, NBUMRL, HNBUE(3), DVRL, DHMRLA, NBUHMRL, HNBUES, HNBUE(2), GNBARFR, NBARFR(3), EBUCA(2), EBUA(2), UBACT, EBU(2), UBAE, EBUCA, HUBAEUT,HNBRUE, RNBRUE, NBUC

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