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Application of Singularly Perturbed Delay Differential Equations in Intuitionistic Fuzzy MAGDM Problems

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Abstract-In this paper, the Multiple Attribute Group Decision Making (MAGDM) problems with Intuitionistic Fuzzy Sets (IFSs) are investigated where the weights of the decision makers about the behaviour of attributes are provided in the form of Singularly Perturbed Delay Differential Equations (SPDDEs). The Decision maker weights are computed using the numerical solutions with a finite difference scheme on Shishkin mesh of SPDDEs. The Intuitionistic Fuzzy Ordered Weighted Averaging (IFOWA) and the Intuitionistic Fuzzy Hybrid Averaging (IFHA) operators are used to aggregate the intuitionistic fuzzy decision matrices. A new correlation coefficient of IFS is proposed and utilised to rank the best alternative. Feasibility and effectiveness of the proposed method is supported with a numerical illustration. Comparison of the proposed model is made with some of the existing distance measures, similarity measures and score functions of ranking the alternatives.

Index Terms-MAGDM; Intuitionistic Fuzzy Sets; OWA operator; Singular perturbation problem; Delay differential equation; Shishkin mesh; Finite difference scheme.

1. INTRODUCTION

In the year 1965, Zadeh [50] introduced the concept of fuzzy set. It contains only a membership function. Later Atanassov [1], [2] and [3] introduced the new type of fuzzy set named as Intuitionistic Fuzzy Sets (IFSs) which is the extension of the concept of fuzzy set. IFSs contains both the membership function and non-membership function. For decision making problems, Szmidt & Kacprzyk [40-44] found the distance between intuitionistic fuzzy sets. The relationship between intuitionistic fuzzy sets, L-fuzzy sets, interval-valued fuzzy sets and interval-valued intuitionistic fuzzy sets were established by Deschrijver & Kerre [10]. For different higher order intuitionistic fuzzy sets, the correlation coefficient was proposed and discussed by Robinson & Amirtharaj [31] to [36] and Robinson [37] and also utilized the correlation measures of ranking the best alternatives in the decision making problem. Yager [49] proposed the ordered weighted averaging (OWA) operator by giving some weights to all inputs according to their ranking positions. Based on its pioneer work, many extensions have been appearing over it to solve the problems of multi-criteria decision making problems. Some geometric aggregation operators for intuitionistic fuzzy sets was proposed and developed by Xu & Yager [48]. Xu also developed some arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, and the intuitionistic fuzzy hybrid aggregation (IFHA) operator in [46] and [47]. Robinson & Jeeva [38] proposed and discussed the various applications of numerical methods in MAGDM problems especially in situations where the weights of the decision makers are completely unpredictable.

Singularly Perturbed Problems or Singular Perturbation Problems (SPPs) always play a prominent role in the theory of differential equations and in their applications to the physical world. Singularly Perturbed Differential Equations (SPDEs) are characterized by a small positive parameter ε multiplying the highest order derivative and/or the lower order derivatives of the differential equations. In [12], [19], [22], [23], [25] and [28], applications of SPP can be extensively observed. A SPP is said to be convection diffusion type if the order of the differential equation is reduced by one when the perturbation parameter ε is set equal to zero. If the order reduces by two, then it is known as reaction diffusion type problem. Miller et.al.in [22] and [23] have devoted their work towards fitted mesh methods and their parameter uniform convergence have also been established. Malley [21] & Nayfeh [26] gave an introduction to singular perturbation problems. Manikandan et.al. [20] presented a numerical method composed of a classical finite difference scheme applied on a piecewise-uniform Shishkin mesh is suggested to solve the singularly perturbed delay differential problem. O'Riordan [29] discussed the presence of any interior layer typically requires the introduction of a transformation of the problem, which facilitates the necessary alignment of the mesh to the trajectory of the interior layer. For discontinuous source term, De Falco & O'Riordan [9] were proposed a uniformly convergent scheme for a system of two coupled singularly perturbed reaction-diffusion robin type and mixed boundary value problems.

Literature Survey on Delay Differential Equations

Extensive works on asymptotic and numerical methods for second order singularly perturbed delay differential problems are reported in the literature. In [5], a class of delay differential equations with a perturbation parameter ε is examined. A hybrid finite difference scheme on an appropriate piecewise uniform mesh of Shishkin-type is derived in [6]. In [16] and [17], boundary value problems for second order singularly perturbed delay differential equations are treated. In [20], a boundary value problem for a second-order singularly perturbed delay differential equation is considered. The singular perturbation theory in ordinary differential equations is extended to delay differential equations with a fixed lag is discussed in [45]. The study in [11] is devoted to the numerical study of boundary value problems for singularly perturbed linear second-order differential-difference equations with a turning point. In [27], the solution to a singularly perturbed second order differential equation with a constant delay having finite element approximation method is studied. In [30], a numerical finite difference method to solve the boundary-value problem for singularly perturbed differential-difference equation, which contains only negative shifts in the differentiated term. In [39], they have studied a numerical solution of singularly perturbed differential-difference equations exhibiting dual layer behaviour. In [13] and [14], computational methods are presented for solving singularly perturbed delay differential equations with negative shift whose solution has boundary layer. In [18], a system of singularly perturbed ordinary differential equations of first order with given initial condition is considered. In [4], a parameter fitted scheme is proposed to solve singularly perturbed delay differential equations of second order with left and right boundary. In [16], a numerical method is proposed to solve boundary-value problems for singularly perturbed differential-difference equations with negative shift. A uniformly-convergent second order Richardson extrapolation technique for solving singularly perturbed delay differential equation is proposed in [24].

In this paper, the decision maker presents his/her attribute weights in the form of a SPDDE. The SPDDEs are solved through numerical methods with a newly proposed finite difference scheme on Shishkin mesh. There are three decision makers in this work whose weights are determined and normalized and utilized in decision making problems. We have investigated the MAGDM problem with intuitionistic fuzzy set for ranking the alternatives together with IFOWA and IFHA operators. Correlation coefficient of IFSs in the range [0,1] is proposed and utilised for ranking the alternatives. A numerical illustration is given to show the effectiveness of the proposed approach.

2. PRELIMINARIES

In this Section, some basic concepts about the IFSs and different classes of aggregation operators are presented.

DEFINITION: Intuitionistic Fuzzy Set, IFS [1-3]

An IFS A in X is given by $A=\left\{\left\langle x,\ \mu_A(x),\ \gamma_A(x)\right\rangle / x\in X\right\}$, where $\mu_A: X\to[0,1], \gamma_A: X\to[0,1]$, with the condition $0\le \mu_A(x)+\gamma_A(x)\le 1,\ \ \forall x\in X$. The numbers $\mu_A(x)$ and $\gamma_A(x)$ represent, the membership degree and non-membership degree of the element x to the set A, respectively.

DEFINITION: Hesitancy Degree of an IFS

For each IFS A in X, if $\pi_A(x)=1-\mu_A(x)-\gamma_A(x),\ \ \forall x\in X$, then $\pi_A(x)$ is called the degree of indeterminacy or hesitancy of x to A, where $0\le\pi_A(x)\le 1$, for all $x\in X$.

2.1. *Different classes of aggregation operators*

Different types of aggregation operators are found in the literature for aggregating the information. Different families of *OWA* operators can be used by choosing a different manifestation of the weighting vector which in detail is discussed in [46], [47] and [49].

DEFINITION: Intuitionistic Fuzzy Weighted Averaging (*IFWA*) operator

Let $\tilde{a}_j=\left(\mu_j,\gamma_j\right)$, for all j = 1,2,…,n be a collection of intuitionistic fuzzy values. The Intuitionistic Fuzzy Weighted Averaging (*IFWA*) operator, $IFWA:\ Q^n\to\ Q$ is defined as:

$$IFWA_\omega\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)$$
$$=\sum_{j=1}^{n}\omega_j\tilde{a}_j=\left(1-\prod_{j=1}^{n}(1-\mu_j)^{\omega_j},\prod_{j=1}^{n}\gamma_j^{\omega_j}\right)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $\tilde{a}_j$, for all j = 1,2,…,n such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1.$

DEFINITION: Intuitionistic Fuzzy Ordered Weighted Averaging (*IFOWA*) operator

Let $\tilde{a}_j = (\mu_j, \gamma_j)$, for all j = 1,2,…,n be a collection of intuitionistic fuzzy values. The Intuitionistic Fuzzy Ordered Weighted Averaging (*IFOWA*) operator $IFOWA\colon Q^n \to Q$ is defined as:

$$IFOWA_w(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$$
$$= \sum_{j=1}^{n} w_j \tilde{a}_{\sigma(j)} = \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{n} \gamma_{\sigma(j)}^{w_j}\right)$$

Where $w = (w_1, w_2, ..., w_n)^T$ is the associated weighting vector such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore, $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of (1,2,…,n), such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for all j = 2,…,n.

DEFINITION: Intuitionistic Fuzzy Hybrid Averaging (IFHA) Operator

Let $a_j = (\mu_j, \gamma_j)$, for all $j = 1, 2, ..., n$ be a collection of intuitionistic fuzzy values. The Intuitionistic Fuzzy Hybrid Aggregation (IFHA) operator, $IFHA : Q^n \to Q$ is defined as:

$$IFHA_{\omega,w}(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} a_{\sigma(j)} w_j$$
$$= \left[1 - \prod_{j=1}^{n} (1 - \mu_{a_{\sigma(j)}})^{w_j}, \prod_{j=1}^{n} (\gamma_{a_{\sigma(j)}})^{w_j}\right],$$

where $w = (w_1, w_2, ..., w_n)^T$ is the associated vector such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of a_j, for all $j = 1, 2, ..., n$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore $a_{\sigma(j)}$ is the j^{th} largest of the weighted IFNs $a_j = a_j^{\,n\omega_j}$, $j = 1, 2, ..., n.$

3. CORRELATION COEFFICIENT OF INTUITIONISTIC FUZZY SETS(IFSs)

In this paper, we propose a new method for calculating correlation coefficient of IFSs based on the method proposed by Robinson & Amirtharaj [31] for calculating correlation coefficient of vague sets, taking not only the membership and non-membership grades into account but also the negation of non-membership degree and hesitancy degree also. Let $X = \{x_1, x_2, ..., x_n\}$ be the finite universal set and let $A, B \in IFS(X)$, be given by

$A = \{\langle x, \mu_A(x_i), \gamma_A(x_i), \pi_A(x_i)\rangle / x \in X\}$,

$B = \{\langle x, \mu_B(x_i), \gamma_B(x_i), \pi_B(x_i)\rangle / x \in X\}$.

The correlation of $A, B \in IFS(X)$ is defined as follows:

$$C_{IFS}(A,B) = \frac{1}{n} \sum_{i=1}^{n} \begin{bmatrix} \mu_A(x_i)\mu_B(x_i) + (1 - \gamma_A(x_i)) \\ (1 - \gamma_B(x_i)) + \pi_A(x_i)\pi_B(x_i) \end{bmatrix}$$

And the correlation coefficient of $A, B \in IFS(X)$ is defined as follows:

$$K_{IFS}(A,B) = \frac{C_{IFS}(A,B)}{\sqrt{C_{IFS}(A,A) C_{IFS}(B,B)}}$$

The following proposition and theorems are true for the above defined correlation coefficient.

Proposition: For $A, B \in IFS(X)$, we have

i) $0 \leq C_{IFS}(A,B) \leq 1$.

ii) $C_{IFS}(A,B) = C_{IFS}(B,A)$.

iii) $K_{IFS}(A,B) = K_{IFS}(B,A)$.

Theorem 3.1. *For* $A, B \in IFS(X)$, *then* $0 \leq K_{IFS}(A,B) \leq 1$.

Theorem 3.2. $K_{IFS}(A,B) = 1 \Leftrightarrow A = B$.

Theorem 3.3. $C_{IFS}(A,B) = 0 \Leftrightarrow$ *A and B are non-fuzzy sets and satisfy the condition* $\mu_A(x_i) + \mu_B(x_i) = 1$ *or* $\gamma_A(x_i) + \gamma_B(x_i) = 1$ *or* $\pi_A(x_i) + \pi_B(x_i) = 1$, $\forall x_i \in X$.

Theorem 3.4. $C_{IFS}(A,A) = 1 \Leftrightarrow$ *A is a non-fuzzy set.*

4. SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS (SPDDEs)

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term. Delay Differential Equations (DDEs) with constant lags $\tau_j > 0$ for j = 1,2,…,k have the form: $y'(t) = f(t, y(t), y(t-\tau_1), ..., y(t-\tau_k))$. An initial value $y(0) = \phi(0)$ is not enough to define a unique solution of $y'(t) = f(t, y(t), y(t-\tau_1), ..., y(t-\tau_k))$ on an interval $a \le t \le b$.The function $y(t) = \phi(t)$ must be specified for $t \le a$ so that $y(t-\tau_j)$ is defined when $a \le t \le a+\tau_j$. The function $\phi(t)$ is called the history of the solution.

In this section, a novel numerical method is suggested to solve a second-order SPDDE of reaction-diffusion type. A hybrid finite difference scheme with an appropriate piecewise uniform Shishkin-type mesh is derived. It is shown that the method is almost second order convergence in the maximum norm, independent of the perturbation parameter.

$Lu(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x)$ on (0,2) (1)

With $u = \varphi$ on $[-1,0]$ and $u(2) = l$ (2)

Where φ is sufficiently smooth on [-1,0]. For all $x \in [0,2]$, it is assumed that $a(x)$ and $b(x)$ satisfy, $a(x) + b(x) > 2\alpha$ and $b(x) < 0$, for some real number $\alpha > 0$. Furthermore, the functions $a(x)$, $b(x)$ and $f(x)$ are assumed to be in $C^3([0,2])$.

The above assumptions ensure that $u \in C = C^0([0,2]) \cap C^1((0,2)) \cap C^2((0,1) \cup (1,2))$.

The problem (1) and (2) can be rewritten as

$$L_1 u(x) = -\varepsilon u''(x) + a(x)u(x)$$
$$= f(x) - b(x)\phi(x-1) = g(x) \text{ on } (0,1).$$
$$L_2 u(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1)$$
$$= f(x) \quad \text{on } (1,2)$$
$$u = \varphi \text{ on } [-1,0],\ u(1-) = u(1+),\ u'(1-) = u'(1+),$$
and $u(2) = l.$

The reduced problem corresponding to (1) and (2) is defined by

$$a(x)u_0(x) = g(x) \text{ on } (0,1)$$
$$a(x)u_0(x) + b(x)u_0(x-1) = f(x) \text{ on } (1,2).$$

4.1. *Shishkin mesh*

In comparison with that of uniform mesh, in piecewise-uniform mesh we give more mesh points in the inner domains so that the layer information can be obtained. A piecewise-uniform Shishkin mesh with N mesh intervals is now constructed on $\bar{\Omega} = [0,2]$ as follows:

Let $\Omega^N = \Omega_1{}^N \cup \Omega_2{}^N$ where

$\Omega_1{}^N = \{x_j\}_{j=1}^{\frac{N}{2}-1}$, $\Omega_2{}^N = \{x_j\}_{j=\frac{N}{2}+1}^{N-1}$ and $x_{\frac{N}{2}} = 1\cdot$

Then $\bar{\Omega}_1 = \{x_j\}_{j=0}^{\frac{N}{2}}$, $\bar{\Omega}_2 = \{x_j\}_{j=\frac{N}{2}}^{N}$,

$$\bar{\Omega}^N = \bar{\Omega}_1{}^N \cup \bar{\Omega}_2{}^N = \{x_j\}_{j=0}^{N}, \text{ and } \Gamma^N = \{0,2\}.$$

The interval [0,1] is divided into three subintervals as follows: $[0,\tau] \cup (\tau, 1-\tau] \cup (1-\tau, 1]$.

The parameter τ, which determine the points separating the uniform meshes, is defined by

$$\tau = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N\right\}.$$

Then, on the sub-interval $(\tau, 1-\tau]$, a uniform mesh with N/4 mesh points is placed and on each of the subintervals $[0,\tau]$ and $(1-\tau, 1]$, a uniform mesh of N/8 mesh points is placed.

Similarly, the interval (1,2] is also divided into 3 subintervals $(1, 1+\tau], (1+\tau, 2-\tau]$ and $(2-\tau, 2]$, using the same parameter τ. In particular, when the parameter τ takes on its left-hand value, the Shishkin mesh $\bar{\Omega}^N$ becomes a classical uniform mesh throughout from 0 to 2.In practice, it is convenient to take N=8k, $k \ge 2$.

4.2. *Discrete problem*

In this section, a hybrid finite difference scheme operator with an appropriate Shishkin mesh is used to construct a numerical method for (1) and (2).The discrete two-point boundary value problem is now defined to be:

$$L^N U(x_j) = -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j) +$$
$$b(x_j)U(x_j - 1) = f(x_j), \text{ on } \Omega^N,\ U = u \text{ on } \Gamma^N.$$

The above problem can be rewritten as

$$L_1{}^N U(x_j) = -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j)$$
$$= g(x_j), \text{ on } \Omega^{-N},$$
$$L_2{}^N U(x_j) = -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j)$$
$$= g(x_j), \text{ on } \Omega^{+N},$$
$$U = u \text{ on } \Gamma,\ D_0^- U(x_{N/2}) = D_0^+ U(x_{N/2}).$$

Where $\delta^2 U(x_j) = \left(\frac{D^+ - D^-}{\bar{h}}\right) U(x_j),$

$$\bar{h} = \frac{h_j + h_{j+1}}{2}.$$

$$D^+ U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{h_{j+1}}; D^- U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j}.$$

The Novel Hybrid Scheme for SPDDEs is given as:

$$D_0^+U(x_j)=\frac{3U(x_{j+1})-4U(x_j)+U(x_{j-1})}{2h_{j+1}},$$
$$D_0^-U(x_j)=\frac{3U(x_j)-4U(x_{j-1})+U(x_{j-2})}{2h_j}$$

This is used to compute numerical approximations to the solution of (1) and (2).

The following discrete results are analogous to those for the continuous case.

Lemma 4.2.1. *Let conditions* $a(x)+b(x)>2\alpha$ *and* $b(x)<0$ *hold. Then, for any mesh function* ψ, *the inequalities* $\psi\ge 0$ on Γ^N, $L_1^N\psi\ge 0$ on Ω_1^N, $L_2^N\psi\ge 0$ on Ω_2^N, *and* $D^+\psi(x_{N/2})-D^-\psi(x_{N/2})\le 0$ *imply that* $\psi\ge 0$ *on* $\bar{\Omega}^N$.

An immediate consequence of this is the following discrete stability result.

Lemma 4.2.2. *Let conditions* $a(x)+b(x)>2\alpha$ *and* $b(x)<0$ *hold. Then, for any mesh function* ψ,

$$|\psi(x_j)|\le \max\left\{ |\psi(x_0)|, |\psi(x_N)|, \frac{1}{\alpha}\|L_1^N\psi\|_{\Omega_1^N}, \frac{1}{\alpha}\|L_2^N\psi\|_{\Omega_2^N} \right\}, 0\le j\le N.$$

5. DETERMING EXPERTS WEIGHTS FOR MAGDM PROBLEMS USING SPDDEs

Problem proposed by the decision maker-1:

The decision maker represents weighting vector about the behaviour of the attributes in the form of the following Singularly Perturbed delay differential Equation:

$-\varepsilon u''(x)+4u(x)-u(x-1)=1+x$ for $x\in(0,2)$,

$u(x)=x^2$ for $x\in[-1,0]$, $u(2)=0$.

Four distinct points based on the parameter τ are identified and the numerical solutions at those points are chosen and normalized for obtaining the weighting vector. Details of the weight vector from the decision maker-1 are given in the following table.

Table 1. Numerical Solution of $-\varepsilon u''(x)+4u(x)-u(x-1)=1+x$

X	Values of U(X)	Weight vector
0.2500000	0.4534445	0.190232379
0.7500000	0.4536239	0.190307642
1.2500000	0.6758255	0.283527296
1.7500000	0.8007408	0.335932683

Hence, the weighting vector given by the first decision maker is calculated as

$$\omega=\begin{pmatrix}0.190232379, 0.190307642,\\ 0.283527296, 0.335932683\end{pmatrix}^T.$$

The maximum point wise errors and the rate of convergence are calculated using the two mesh algorithm in [12] and are presented in the following table:

Table 2. Values of $D_\varepsilon^N, D^N, p^N, C_p^N$ for $\varepsilon=\frac{\eta}{4}$ and α=0.9.

η	128	256	512	1024
2^{-2}	0.674E-03	0.336E-03	0.168E-03	0.840E-04
2^{-3}	0.921E-03	0.461E-03	0.230E-03	0.115E-03
2^{-4}	0.128E-02	0.642E-03	0.320E-03	0.160E-03
2^{-5}	0.180E-02	0.907E-03	0.453E-03	0.226E-03
2^{-6}	0.250E-02	0.128E-02	0.643E-03	0.321E-03
D^N	0.250E-02	0.128E-02	0.643E-03	0.321E-03
p^N	0.961E+00	0.999E+00	0.100E+01	
D_p^N	0.544E+00	0.544E+00	0.529E+00	0.514E+00

The order of convergence=0.9605488E+00.
The error constant=0.5436762E+00.
It can be noted that as N increases and η decreases, the error decreases. Numerical solution of $-\varepsilon u''(x)+4u(x)-u(x-1)=1+x$ is displayed in figure 1.

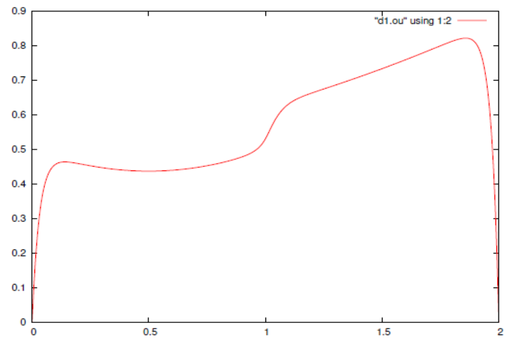

Figure 1. Numerical solution of $-\varepsilon u''(x)+4u(x)-u(x-1)=1+x$

Problem proposed by the decision maker-2:

The decision maker represents weighting vector about the behaviour of the attributes in the form of the following Singularly Perturbed delay differential Equation:

$$-\varepsilon u''(x)+4u(x)-u(x-1)=2 \text{ for } x\in(0,2),$$
$$u(x)=x^2 \text{ where } x\in[-1,0],\ u(2)=0.$$

Four distinct points based on the parameter τ are identified and the numerical solutions at those points are chosen and normalized for obtaining the weighting vector. Details of the weight vector from the decision maker-2 are given in the following table.

Table 3. Numerical Solution of $-\varepsilon u''(x)+4u(x)-u(x-1)=2$

X	Values of U(X)	Weight vector
0.2500000	0.6408602	0.261998543
0.7500000	0.5161290	0.211005530
1.2500000	0.6601004	0.269864384
1.7500000	0.6289553	0.257131543

Hence, the weighting vector given by the first decision maker is calculated as

$$w=\begin{pmatrix}0.261998543, 0.211005530,\\ 0.269864384, 0.257131543\end{pmatrix}^T.$$

The maximum point wise errors and the rate of convergence are calculated using the two mesh algorithm in [12] and are presented in table 4:

Table 4. Values of $D_\varepsilon^N, D^N, p^N, C_p^N$ for $\varepsilon=\frac{\eta}{4}$ and α=0.9.

η	128	256	512	1024
2^{-2}	0.912E-03	0.458E-03	0.229E-03	0.115E-03
2^{-3}	0.130E-02	0.653E-03	0.325E-03	0.162E-03
2^{-4}	0.184E-02	0.927E-03	0.463E-03	0.231E-03
2^{-5}	0.262E-02	0.133E-02	0.663E-03	0.331E-03
2^{-6}	0.368E-02	0.190E-02	0.950E-03	0.474E-03
D^N	0.368E-02	0.190E-02	0.950E-03	0.474E-03
p^N	0.957E+00	0.997E+00	0.100E+01	
D_p^N	0.789E+00	0.789E+00	0.768E+00	0.743E+00

The order of convergence=0.9571305E+00.
The error constant=0.7893250E+00.

It can be noted that as N increases and η decreases, the error decreases. Numerical solution of $-\varepsilon u''(x)+4u(x)-u(x-1)=2$ is displayed in figure 2.

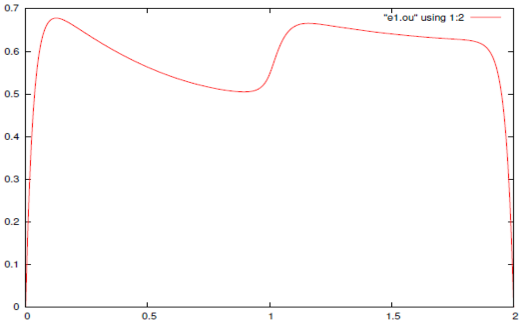

Figure 2. Numerical solution of $-\varepsilon u''(x)+4u(x)-u(x-1)=2$

Problem proposed by the decision maker-3:

The decision maker represents weighting vector about the behaviour of the attributes in the form of the following Singularly Perturbed delay differential Equation:

$$-\varepsilon u''(x)+(2+x)u(x)-u(x-1)=0 \text{ for } x\in(0,2),$$
$$u(x)=1 \text{ for } x\in[-1,0],\ u(2)=1.$$

Four distinct points based on the parameter τ are identified and the numerical solutions at those points are chosen and normalized for obtaining the weighting vector. Details of the weight vector from the decision maker-3 are given in the following table.

Table 5. Numerical Solution of $-\varepsilon u''(x)+(2+x)u(x)-u(x-1)=0$

X	Values of U(X)	Weight vector
0.2500000	0.4461796	0.426800856
0.7500000	0.3637271	0.347929484
1.2500000	0.1381052	0.132106931
1.7500000	0.09739275	0.093162729

Hence, the weighting vector given by the third decision maker is calculated as

$$\gamma=\begin{pmatrix}0.426800856, 0.347929484,\\ 0.132106931, 0.093162729\end{pmatrix}^T.$$

The maximum point wise errors and the rate of convergence are calculated using the two mesh algorithm in [12] and are presented in table 6:

Table 6. Values of $D_\varepsilon^N, D^N, p^N, C_p^N$ for $\varepsilon=\frac{\eta}{4}$ and α=0.9

η	128	256	512	1024
2^{-2}	0.694E-03	0.344E-03	0.171E-03	0.854E-04
2^{-3}	0.103E-02	0.507E-03	0.252E-03	0.125E-03
2^{-4}	0.146E-02	0.719E-03	0.356E-03	0.177E-03
2^{-5}	0.207E-02	0.102E-02	0.502E-03	0.249E-03

2^{-6}	0.291E-02	0.144E-02	0.708E-03	0.350E-03
D^N	0.291E-02	0.144E-02	0.708E-03	0.350E-03
p^N	0.102E+01	0.102E+01	0.101E+01	
D_p^N	0.794E+00	0.792E+00	0.787E+00	0.787E+00

The order of convergence=0.1014829E+01.
The error constant=0.7937517E+00.
It can be noted that as N increases and η decreases, the error decreases. Numerical solution of $-\varepsilon u''(x)+(2+x)u(x)-u(x-1)=0$ is displayed in figure 3.

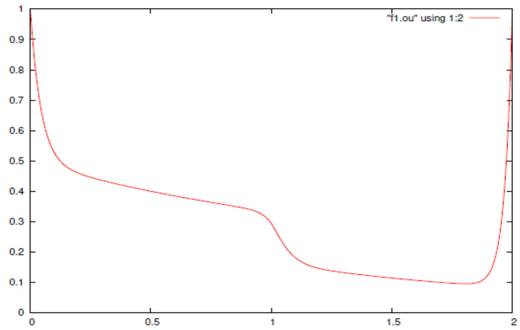

Figure 3. Numerical solution of $-\varepsilon u''(x)+(2+x)u(x)-u(x-1)=0$

6. ALGORTHIM FOR GROUP DECISION MAKING WITH INTUITIONISTIC FUZZY INFORMATION

Step:1 Use the IFOWA operator to aggregate all individual intuitionistic fuzzy decision matrices $R^{(k)}=\left(r_{ij}^{(k)}\right)_{m\times n}$ $(k=1,2,3,4)$ into a collective intuitionistic fuzzy decision matrix $R=(r_{ij})_{mxn}$.

Step:2 Utilize the IFHA operator, $\tilde{r}_i=(\mu_i,\gamma_i)=IFHA_{v,w}=\left(\tilde{r}_i^{(1)},\tilde{r}_i^{(2)},\ldots,\tilde{r}_i^{(t)}\right), i=1,2,\ldots m$ to derive the collective overall preference intuitionistic fuzzy values $\tilde{r}_i\left(i=1,2,\ldots..m\right)$ of the alternative A_i, with the weighting vectors of the decision makers.

Step:3 Calculate the correlation between the collective overall preference values r_i and the positive ideal value $\tilde{r}_i$, where $\tilde{r}_i=(0,1)$.

$$C_{IFS}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left[\begin{matrix}u_A(x_i)u_B(x_i)+(1-\gamma_A(x_i))\\(1-\gamma_B(x_i))+\pi_A(x_i)\pi_B(x_i)\end{matrix}\right]$$

Step:4 Calculate the correlation coefficient between the collective overall preference values r_i and the positive ideal value $\tilde{r}_i$, where $\tilde{r}_i=(0,1)$.

$$K_{IFS}(A,B)=\frac{C_{IFS}(A,B)}{\sqrt{C_{IFS}(A,A)C_{IFS}(B,B)}}.$$

Step:5 Rank all the alternatives $A_i\left(i=1,2,\ldots,m\right)$ and select the best one.

7. NUMERICAL ILLUSTRATION

Suppose that a tele-communication company intends to choose a manager for Research and Development department from five volunteers. The decision making committee is assessing the five concerned volunteers based on four attributes shown as follows:
C1-Proficiency in identifying research areas;
C2-Proficiency in administration;
C3-Personality;
C4-Self Confidence.
The five possible alternatives $A_i\left(i=1,2,3,4,5\right)$ are to be evaluated using intuitionistic fuzzy numbers by the three decision makers whose weighting vectors are calculated using singularly perturbed delay differential equation. The decision information is listed in the following matrices as follows:

$$R_1=\begin{bmatrix}(0.5121,0.3239)&(0.5232,0.3421)&(0.3421,0.5213)&(0.2142,0.6124)\\(0.6215,0.3128)&(0.4852,0.3241)&(0.6421,0.2213)&(0.4164,0.4315)\\(0.5213,0.3168)&(0.5285,0.3512)&(0.5212,0.3582)&(0.2685,0.3585)\\(0.7125,0.1625)&(0.6235,0.2285)&(0.3543,0.4165)&(0.2635,0.5285)\\(0.5215,0.2358)&(0.4586,0.2385)&(0.6212,0.2168)&(0.1892,0.3562)\end{bmatrix}$$

$$R_2=\begin{bmatrix}(0.4562,0.3283)&(0.5283,0.2652)&(0.2125,0.5165)&(0.1285,0.6256)\\(0.5162,0.2856)&(0.5131,0.2165)&(0.4215,0.3865)&(0.3583,0.4265)\\(0.5281,0.3165)&(0.4123,0.3369)&(0.4465,0.2315)&(0.5213,0.2260)\\(0.6310,0.2215)&(0.5560,0.2131)&(0.2650,0.3326)&(0.1100,0.5002)\\(0.5161,0.1100)&(0.3265,0.2613)&(0.6125,0.2010)&(0.4010,0.2165)\end{bmatrix}$$

$$R_3=\begin{bmatrix}(0.4422,0.3366)&(0.5232,0.4100)&(0.2565,0.6213)&(0.1122,0.6180)\\(0.5215,0.3316)&(0.5162,0.3322)&(0.5283,0.3625)&(0.3165,0.5500)\\(0.4422,0.4422)&(0.4265,0.4125)&(0.4000,0.4263)&(0.5213,0.3311)\\(0.6215,0.2233)&(0.4200,0.3921)&(0.2581,0.5110)&(0.1265,0.6001)\\(0.5126,0.3163)&(0.3366,0.4410)&(0.6123,0.2121)&(0.4111,0.4211)\end{bmatrix}$$

$$R_4=\begin{bmatrix}(0.1283,0.6211)&(0.3168,0.5162)&(0.3003,0.5110)&(0.2650,0.4125)\\(0.5120,0.2121)&(0.4216,0.2213)&(0.5168,0.2650)&(0.3256,0.4251)\\(0.2683,0.3333)&(0.4466,0.3612)&(0.3440,0.2862)&(0.2168,0.2656)\\(0.2666,0.5123)&(0.6600,0.1111)&(0.2220,0.5012)&(0.5161,0.2300)\\(0.7100,0.1220)&(0.2255,0.3660)&(0.6121,0.1131)&(0.4220,0.3413)\end{bmatrix}$$

By using, step 1 and step 2 of the proposed algorithm, we get the overall values as:
$\tilde{r}_1 = [0.408534857357508, 0.477653955584503]$;
$\tilde{r}_2 = [0.478623907908194, 0.345049520011432]$;
$\tilde{r}_3 = [0.467947992690850, 0.340830269503761]$;
$\tilde{r}_4 = [0.393939877495897, 0.352536594030511]$;
$\tilde{r}_5 = [0.425384973944518, 0.259275986154004]$.
By using step 3 and step 4 of the algorithm, we calculate the correlation coefficient as follows:
$K(\tilde{r}_1, \tilde{r}^+)$ = 0.978305069439398,
$K(\tilde{r}_2, \tilde{r}^+)$ = 0.965570095974722,
$K(\tilde{r}_3, \tilde{r}^+)$ = 0.959434473942105,
$K(\tilde{r}_4, \tilde{r}^+)$ = 0.921435879554324,
$K(\tilde{r}_5, \tilde{r}^+)$ = 0.905587331782648.

Ranking the best alternative according to the correlation coefficient, we get:
$A_1 > A_2 > A_3 > A_4 > A_5$.
Using step 3 of the same algorithm proposed in the paper with different distance measures [15], we obtain the ranking of best alternatives and the results are tabulated as follows:

Table 7. Comparison of the Ranking with Distance Measures

DISTANCE MEASURES	RANKING THE ALTERNATIVES
$d_1(A,B)$. Hamming Distance	$A_1 > A_4 > A_3 > A_2 > A_5$
$d_2(A,B)$. Normalized Hamming Distance	$A_1 > A_4 > A_3 > A_2 > A_5$
$d_3(A,B)$. Euclidean Distance	$A_1 > A_4 > A_3 > A_5 > A_2$
$d_4(A,B)$. Normalized Euclidean Distance	$A_1 > A_4 > A_3 > A_5 > A_2$
$d_5(A,B)$. Hamming Distance	$A_4 > A_1 > A_4 > A_3 > A_2$
$d_6(A,B)$. Normalized Hamming Distance	$A_1 > A_4 > A_3 > A_5 > A_2$
$d_7(A,B)$. Euclidean Distance	$A_1 > A_4 > A_5 > A_3 > A_2$
$d_8(A,B)$. Normalized Euclidean Distance	$A_1 > A_4 > A_5 > A_3 > A_2$
$d_{h1}(A,B)$. Hamming Distance	$A_4 > A_1 > A_4 > A_3 > A_2$
$d_{h2}(A,B)$. Normalized Hamming Distance	$A_4 > A_1 > A_4 > A_3 > A_2$
$d_{h3}(A,B)$. Euclidean Distance	$A_4 > A_1 > A_5 > A_2 > A_3$
$d_{h4}(A,B)$. Normalized Euclidean Distance	$A_4 > A_1 > A_5 > A_2 > A_3$

Using step 3 of the same algorithm proposed in the paper with different similarity measures [8], we obtain the ranking of best alternatives and the results are tabulated as follows:

Table 8. Comparison of the Ranking with Similarity Measures

SIMILARITY MEASURES	RANKING THE ALTERNATIVES
1.Chen's Measure	$A_5 > A_2 > A_3 > A_4 > A_1$
2.Hong &Kim's Measure	$A_5 > A_2 > A_3 > A_4 > A_1$
3.Li &Xu'sMeasure	$A_5 > A_2 > A_3 > A_4 > A_1$
4.Li et al.'s Measure	$A_5 > A_2 > A_3 > A_4 > A_1$
5.Li &Chuntian's Measure	$A_5 > A_2 > A_3 > A_4 > A_1$
6.Mitchell's Measure	$A_5 > A_3 > A_2 > A_4 > A_1$
7.Hung & Yang's Measure (SHY_1) SHY_2 SHY_3	$A_5 > A_3 > A_2 > A_4 > A_1$ $A_5 > A_3 > A_2 > A_4 > A_1$ $A_5 > A_3 > A_2 > A_4 > A_1$
8.Ye's Measure	$A_5 > A_2 > A_3 > A_4 > A_1$
9.Chen, S.M., & Randyanto,Y,. Measure	$A_2 > A_3 > A_5 > A_4 > A_1$

Using step 3 of the same algorithm proposed in the paper with different score functions [7], we obtain the ranking of best alternatives and the results are tabulated as follows:

Table 9. Comparison of the Ranking with Score Functions

SCORE FUNCTIONS	RANKING THE ALTERATIVES
1. $S_1(X_{ij})$	$A_5 > A_2 > A_3 > A_4 > A_1$
2. $S_2(X_{ij})$	$A_2 > A_3 > A_1 > A_5 > A_4$
3. $S_3(X_{ij})$	$A_2 > A_3 > A_5 > A_1 > A_4$

4.	$S_4(X_{ij})$	$A_1 > A_2 > A_3 > A_4 > A_5$
5.	$S_5(X_{ij}),\ \lambda = 0.3$	$A_5 > A_2 > A_3 > A_4 > A_1$
6.	$S_5(X_{ij}),\ \lambda = 0.5$	$A_5 > A_2 > A_3 > A_4 > A_1$
7.	$S_5(X_{ij}),\ \lambda = 0.7$	$A_2 > A_3 > A_5 > A_4 > A_1$

8. DISCUSSION

In this work, the weights of the decision makers are presented in the form of Singularly Perturbed Delay Differential Equations whose solutions are sought through a new hybrid finite difference scheme operator with an appropriate Shishkin mesh which is used to construct numerical methods to solve the SPDDEs. From table 7, 8 and 9 it can be observed that using different ranking methods can yield differences in the ranking of best alternative. Since correlation coefficient can bring out the linear relationship between the variables under study, the ranking of alternatives based on the new proposed correlation coefficient of IFSs in this paper can be considered to be the best ranking process.

9. CONCLUSION

In this work, we have discussed about the weight determining methods together with weighted averaging operator and the ordered weighted averaging operator, which extend two of the most common aggregation operators to accommodate the situations where the input arguments are intuitionistic fuzzy values. The *IFWA* operator weights only the intuitionistic fuzzy arguments, while the *IFOWA* operator weights only the ordered positions of the intuitionistic fuzzy arguments instead of weighting the intuitionistic fuzzy arguments themselves. We have developed the MAGDM model based on hybrid aggregation operator (*IFHA*), which weights both the given interval valued intuitionistic fuzzy value and its ordered position of the argument. In this paper, a new approach for determining weights of decision makers in group decision environment based on singular perturbation problem is proposed. The decision maker weights are calculated using the numerical solution of SPDDE of reaction-diffusion type problem and applied in MAGDM problems under intuitionistic fuzzy set. The weights calculated by the proposed method, which can relieve the influence of unfair arguments on the final results by assigning low weights to those unduly high or unduly low ones, and hence make the decision results more precise and reasonable when applied to decision making based on intuitionistic fuzzy information. Different comparisons were made with existing ranking methods and the effectiveness of the proposed method was displayed by numerical illustration.

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