On Non-Homogeneous Sextic Equation with Five Unknowns

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Abstract- The non-homogeneous sextic equation with five unknowns given by \((2x + 2y)(2x^3 - 2y^3) = 32(2z^2 - 2w^2)T^4\) is considered and analysed for its non-zero distinct integer solutions. Employing the linear transformations \((x, y, z, w, T)\) and applying the method of factorization, three different patterns of non-zero distinct integer solutions are obtained. A few interesting relation between the solutions and special numbers.

Keywords: Integer solutions, Non-homogeneous sextic equations with five unknowns.

1. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4] particularly, in [5, 6] sextic equations with three unknowns are studied for their integral solutions [7, 12] analyze Sextic equations with five unknowns for their non-zero integer solution. This communication analyzes a Sextic equations with five unknowns given by \((2x + 2y)(2x^3 - 2y^3) = 32(2z^2 - 2w^2)T^4\). Infinitely many Quintuples \((x, y, z, w, T)\). Satisfying the above equation is obtained. Various interesting properties among the values of \(x, y, z, w\) and \(T\) are presented.

2. NOTATIONS

- \(t_{m,n}\) = Polygonal number of rank \(n\) with size \(m\)
- \(P_n^m\) = pyramidal number of rank \(n\) with size \(m\)
- \(P_n^3\) = pronic number of rank \(n\)
- \(SO_n\) = Stella octagonal number of rank \(n\)
- \(j_n\) = jacobthalucas number of rank \(n\)
- \(J_n\) = Jacobthal number of rank \(n\)
- \(G_{no}^n\) = Gnomic number of rank \(n\)
- \(C_p^6\) = Centered pyramidal number of rank \(n\) with size \(m\)
- \(C_{p,14}\) = Centered tetra decagonal pyramidal number of rank \(n\)
- \(Ky_n\) = Kynea number of rank \(n\).
- \(Obl_n\) = Oblong Number of rank \(n\)
- \(CH_n\) = Centered Hexagonal number of rank \(n\).
- \(CP_n\) = Centered pentagonal number of rank \(n\).
- \(S_n\) = Star number of rank \(n\).
- \(4DF_n\) = Four Dimensional figurate number whose generating polygon is a square.

3. METHOD OF ANALYSIS

The non-homogeneous sextic equation with five unknowns to be solved is given by.
\[(2x + 2y)(2x^3 - 2y^3) = 32(2z^2 - 2w^2)T^4\]

\[\text{--------(1)}\]

The substitution of the linear transformations \(x = 2u + 2v, y = 2u - 2v, z = 4u + v, w = 4u - v, (u \neq v \neq 0)\) in (1) leads to
\[v^2 + 3u^2 = 4T^4\]

\[\text{--------(2)}\]

\(3)\) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

Pattern:1

Assume \(T = T(a, b) = a^2 + 3b^2, a, b > 0\)

\[\text{--------(4)}\]

Write 4 as
\[4 = (1 + i\sqrt{3})(1 - i\sqrt{3})\]

Using (4) and (5) in (3) and employing the method of factorization and equating positive factors, we get
\[v + i\sqrt{3}u = (1 + i\sqrt{3})(a + i\sqrt{3}b)^4\]

Equating real and imaginary parts, we have
\[u = u(a, b) = a^4 + 9b^4 - 18a^2b^2 + 4a^3b - 12ab^3\]
Employing (2), the values of \(x, y, z, w\) and \(T\) are given by

\[
\begin{align*}
x &= x(a, b) = 4a^4 + 36b^4 - 72a^2b^2 - 16a^3b + 52ab^3 \\
y &= y(a, b) = 32a^3b - 96ab^3 \\
z &= z(a, b) = 5a^4 + 45b^4 - 90a^2b^2 + 4a^3b - 12ab^3 \\
w &= w(a, b) = 3a^4 + 27b^4 - 54a^2b^2 + 28a^3b - 84ab^3 \\
T &= T(a, b) = a^2 + 3b^2
\end{align*}
\]

Which represent non-zero distinct integer solutions of (1) in two parameters.

**Properties**

- \(T(1, 2^n) + 9j_n + 3j_n - 4 = 3kj_n\)
- \(x(a, 1) - 4(T_{4,a})^2 - 16CP_a^6 + 72T_{4,a} \equiv 36 \mod 52\)
- \(w(a, 1) + T(a^2, 1) - 4F_{10,a} + 50T_{4,a} + 2816CP_a^6 - 30 = 0\)
- \(T(a^2, a^2) - 4(T_{4,a})^2 = 0\)
- \(y(1, b) - 3CP_b^6 - 67CP_b^2 + 72CP_b^3 \equiv 0 \mod 72\)
- \(y(a, 1) - z(a, 1) - w(a, 1) + 384DFb - 136Obl_b \equiv 72 \mod 136\)
- \(T(a, a + 1) - T_{4,a} - Pr_a - 3Gna_a = 6\)
- \(z(1, b) - 180DFb - 24P_{25}^5 + CB_b - 36Obl_b \equiv 4 \mod 43\)
- \(7z(1, b) - w(1, b) - 3456FN_b^4 + 2(T_{4,12b}) - J_4 - 3j_b = 0\)
- \(T(b + 1, b + 1) - 12FN_b^4 - 2CP_b - 5Obl_b \equiv 1 \mod 2\)

**Pattern: 2**

One may write (3) as

\[
v^2 + 3u^2 = 4T^4 \cdot 1
\]

Also, write 1 as

\[
1 = \frac{2 + (2\sqrt{3})(2 - 2\sqrt{3})}{16}
\]

Substituting (4), (5) and (7) in (6) and employing the method of factorization and equating positive factors we get

\[
v + i\sqrt{3}u = (1 + i\sqrt{3})\frac{(2 + (2\sqrt{3})(2 - 2\sqrt{3}))}{4}(a + \sqrt{3}b)^4
\]

Equating real and imaginary parts, we have

\[
\begin{align*}
u &= u(a, b) = a^4 + 9b^4 - 18a^2b^2 - 12a^3b + 36ab^3 \\
v &= v(a, b) = -a^4 - 9b^4 + 18a^2b^2 - 12a^3b + 36ab^3
\end{align*}
\]

In view of (2), the integer values of \(x, y, z, w\) and \(T\) are given by

\[
\begin{align*}
x &= x(a, b) = -32a^3b + 96ab^3 \\
y &= y(a, b) = 4a^4 + 36b^4 - 72a^2b^2 + 16a^3b - 48ab^3 \\
z &= z(a, b) = 3a^4 + 27b^4 - 54a^2b^2 - 28a^3b + 84ab^3 \\
w &= w(a, b) = 5a^4 + 45b^4 - 90a^2b^2 - 4a^3b + 12ab^3 \\
T &= T(a, b) = a^2 + 3b^2
\end{align*}
\]

Which represent non-zero distinct integer solutions of (1) in two parameters.

**Properties**

- \(3y(a, 1) - 4z(a, 1) - 480SO_a - 800CP_a^6\)
- \(3x(a, a) + 16T(a^2, a^2)\) is a biquadratic number
- \(w(a, 1) + 90T(a, 1) - 5(T_{4,a})^2 + 8CP_a^6 + 16Gna_a - j_b = 74\)
- \(z(a, 1) - 37T(a^2, 1) + 12CP_a^4 + 54(T_{4,a}) \equiv 18 \mod 68\)
- \(x(1, b) - 36Biq_b - 52Cu_b + 18T_{4,2b} \equiv 4 \mod 16\)
- \(x(1, b) - z(1, b) - w(1, b) + 864FN_b^4 - 2(T_{4,6b}) + j_i = 3\)
- \(T(2a + 1, 2a) - T_{13,a} + 5Obl_a + 4Gna_a + j_b = 74\)
- \(x(b, b + 1) - 384FN_b^4 - 32T_{4,b}^2 - 64Obl_b - 512P_b^5 \equiv 0 \mod 32\)
- \(z(a + 1, a) - 29Biq_a - 105CP_a^6 + 138T_{4,a} - 9Obl_a + j_3 + j_5 = 0\)
- \(w(1, b) - 5T(1, b) - 45T_{4,b}^2 - 18OH_b + 95T_{4,b} + 10Obl_b\)

**Pattern: 3**
Write (3) as
\[3(u^2 - T^4) = T^4 - v^2\]  \hspace{1cm} (10)

Factorizing (10) we have
\[3((u + T^2)(u - T^2)) = (T^2 + v)(T^2 - v)\]  \hspace{1cm} (11)

This equation is written in the form of ratio as
\[
\frac{3(u^2 - T^2)}{T^2 - v} = \frac{u^2 + v^2}{u + v}, \quad b \neq 0
\]  \hspace{1cm} (12)

Which is equivalent to the system of double equations
\[3bu + av - (3b + a)T^2 = 0\]  \hspace{1cm} (13)
\[-au + bv + (b - a)T^2 = 0\]  \hspace{1cm} (14)

Applying the method of cross multiplication, we get
\[u = -a^2 + 3b^2 + 2ab\]  \hspace{1cm} (15)
\[v = a^2 - 3b^2 + 6ab\]  \hspace{1cm} (16)
\[T^2 = a^2 + 3b^2\]  \hspace{1cm} (17)

Now, the solution for (17) is
\[a = 3p^2 - q^2, \quad b = 2pq, \quad T = 3p^2 + q^2\]  \hspace{1cm} (18)

Using (18) in (15) and (16), we get
\[u = u(p, q) = -9b^4 - q^4 + \frac{18p^2q^2}{12p^3q - 4pq^3}\]
\[v = v(p, q) = 9p^4 + q^4 - 18p^2q^2 + 36pq^3\]
\[-12pq^3\]

In view of (2), the integer values of x,y,z,w and T are given by
\[x = x(p, q) = 2u + 2v = 96p^3q - 36pq^3\]
\[y = y(p, q) = 2u - 2v = -36p^4 - 4q^4 + 72p^2q^2 - 48pq^3\]
\[w = w(p, q) = 36pq^3\]
\[T = T(p, q) = 3p^2 + q^2\]

Which represent non-zero distinct integer solutions of (1) in two parameters

Properties

\[x(p, 1) + 2y(p, 1) + 864FN_p^4 - 72T_{4p} + j_3 = 1\]
\[3x(p, p) + 16T(p^2, p^2)\] is a biquadratic number
\[w(p, 1) - 30T(p, 1) + 45(T_{4p})^2 - 450p + 4Cn^6 + 35 = 0\]
\[T(1, 2)\] is mersenne primes and perfect number
\[x(p, 1) - 4BSP_{1p} \equiv 0(\text{mod} 16)\]
\[y(1, q) + 48FN_q^4 - 24OH_q - T_{24q} - 57T_{4q} \equiv 36(\text{mod} 46)\]
\[T(2p, 2p + 2) - 5p - 10pr_{4p} - 26n_{4p} - j_2\]
\[x(1, q) + T(1, q) + 16SO_q - Obq - G_{onq} \equiv 4(\text{mod} 77)\]
\[z(1, q) + 3T_{4q} - 28C_{p^6} - 54P_{1q} - 15G_{onq} + j_3 = 5\]

4. CONCLUSION
First of all, it is worth to mention here that in (2), the values of z and w may also be represented by \(z = uv + 4, \ w = u^2 - v^4 - 4, \ w = 2uv + 2, \ w = 2uv - 2\) and thus, will obtain other choices of solutions to (1). In conclusion, one may consider other forms of Sextic equation with five unknowns and search for their integer solutions.

REFERENCES


