Integral solutions of Quadratic Diophantine equation

$$4w^2 - x^2 - y^2 + z^2 = 16t^2$$

with five unknowns

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Abstract - The Quadratic Diophantine equation given by

$$4w^2 - x^2 - y^2 + z^2 = 16t^2$$

is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Quadratic, integral solutions, polygonal numbers.

INTRODUCTION

The theory of Diophantine equation offers a rich variety of fascinating problems. There are Diophantine problems, which involve quadratic equations with five variables. Quadratic Diophantine equations with five unknowns are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [4-15]. In this communication, we consider yet another interesting quadratic equation

$$4w^2 - x^2 - y^2 + z^2 = 16t^2$$

and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special Polygonal numbers are presented.

OTATIONS USED

- $t_{mn}$ - Polygonal number of rank ‘n’ with size ‘m’
- $C_{kn}^6$ - Centered hexagonal Pyramidal number of rank ‘n’
- $G_{an}$ - Gnomic number of rank ‘n’
- $FN_{kn}$ - Figurative number of rank ‘n’ with size ‘m’
- $Pr_{kn}$ - Pronic number of rank ‘n’
- $P_kn^m$ - Pyramidal number of rank ‘n’ with size ‘m’
- $So_{an}$ - Stella octagonal number of rank ‘n’
- $sn$ - Star number of rank ‘n’
- $j_n$ - Jacobsthal –Lucas number of rank ‘n’
- $T_n$ - Triangular number of rank ‘n’
- $Hex_{kn}$ - Hexagonal number of rank ‘n’
- $Ob_{kn}$ - Oblong number of rank ‘n’
- $OH_n$ - Octagonal number of rank ‘n’
- $PT_n$ - Pentagonal number of rank ‘n’

METHOD OF ANALYSIS

The Quadratic Diophantine equation with five unknowns to be solved for its non-zero distinct integral solutions is

$$4w^2 - x^2 - y^2 + z^2 = 16t^2$$

On substituting the linear transformation

$$x = w + z$$
$$y = w - z$$

in (1), leads to

$$2w^2 - (4t)^2 = z^2$$

We obtain different patterns of integral solutions to (1) through solving (3) which are illustrated as follows

PATTERN I

Equation (3) can be written as

$$w^2 - (4t)^2 = z^2 - w^2$$

$$(w + 4t)(w - 4t) = (z + w)(z - w)$$

Choice 1:

Equation (4) can be written in the form of ratio as

$$w + 4t = \frac{z + w}{A}$$
$$w - 4t = \frac{z - w}{B}$$

Which is equivalent to the system of equations

$$(B - A)w + 4Bt - Az = 0$$
$$(A + B)w - 4At - Bz = 0$$

Applying the method of cross multiplication and simplifying, the values of $w$, $t$ and $z$ are as follows
Substituting (5) in (2), the corresponding non-zero distinct integral solutions of (1) are given by
\[ x(A, B) = x = 8A^2 + 8AB \\
\]
\[ y(A, B) = y = 8B^2 - 8AB \\
\]
\[ z(A, B) = z = 4A^2 + 4B^2 \\
\]
\[ w(A, B) = w = A^2 - B^2 - 2AB \\
\]
\[ t(A, B) = t = 4A^2 + 8AB - 4B^2 \\
\]

Properties
1. \( z(1, B) + w(1, B) - 16t_B \equiv 0 \)
2. \( z(1, B(B + 2)) + w(1, B(B + 2)) - 48B^3 \equiv 0 \)
3. \( x(A, A) + y(A, A) \) can be expressed as a perfect square
4. \( x(1,1) \) is a perfect square
5. \( t(2,1) \) as a carol number
6. \( x(0,1) - t(1,2) \) is a Kynea number
7. \( w(1,1) + t(0,1) \) is a Woodall number
8. The following expression represents a Nasty number
   a. \( x(1,2) - w(0,1) \)
   b. \( x(2,1) \)
   c. \( w(1,2) + z(0,1) \)
9. The following expression represents a perfect number
   a. \( x(1,2) - w(1,2) \)
   b. \( z(1,2) \)

Choice 2:
Equation (4) can be rewritten in the form of ratio as
\[ \frac{w}{z} = \frac{w + 4t}{z - w} = \frac{A}{B} , \quad B \neq 0 \]
Which is equivalent to the system of equations
\[ (A + B)w + 4Bt - Az = 0 \]
\[ (B - A)w + 4At + Bz = 0 \]
Applying the method of cross multiplication and simplifying, the values of w, t and z are as follows
\[ w = w(A, B) = 4A^2 + 4B^2 \]
\[ t = t(A, B) = A^2 - 2AB - B^2 \]
\[ z = z(A, B) = 4A^2 + 8AB - 4B^2 \]
\[ \] Substituting (6) in (2), the corresponding non-zero distinct integral solutions of (1) are given by
\[ x(A, B) = x = 8A^2 + 8AB \\
\]
\[ y(A, B) = y = 8B^2 - 8AB \\
\]
\[ z(A, B) = z = 4A^2 + 4B^2 \\
\]
\[ w(A, B) = w = A^2 - B^2 - 2AB \\
\]
\[ t(A, B) = t = 4A^2 + 8AB - 4B^2 \\
\]

Choice 3:
Equation (4) can be rewritten in the form of ratio as
\[ \frac{w - 4t}{z + w} = \frac{A}{B} , \quad B \neq 0 \]
Which is equivalent to the system of equations
\[ (B - A)w - 4Bt - Az = 0 \]
\[ (-B - A)w - 4At + Bz = 0 \]
Applying the method of cross multiplication and simplifying, the values of w, t and z are as follows
\[ w = w(A, B) = -4A^2 - 4B^2 \]
\[ t = t(A, B) = A^2 + 2AB - B^2 \]
\[ z = z(A, B) = 4A^2 - 8AB - 4B^2 \]
\[ \] Substituting (7) in (2), the corresponding non-zero distinct integral solutions of (1) are given by
\[ x(A, B) = x = -8B^2 - 8AB \\
\]
\[ y(A, B) = y = -8A^2 + 8AB \\
\]
\[ z(A, B) = z = 4A^2 - 4B^2 \\
\]
\[ w(A, B) = w = A^2 - B^2 + 2AB - A^2 \\
\]
\[ t(A, B) = t = -4A^2 - 8AB + 4B^2 \]

Choice 4:
Equation (4) can be rewritten in the form of ratio as
\[ \frac{w - 4t}{z - w} = \frac{A}{B} , \quad B \neq 0 \]
Which is equivalent to the system of equations
\[ (B + A)w - 4Bt - Az = 0 \]
\[ (B - A)w - 4At + Bz = 0 \]
Applying the method of cross multiplication and simplifying, the values of w, t and z are as follows
\[ w = w(A, B) = -4A^2 - 4B^2 \]
\[ t = t(A, B) = -A^2 + 2AB + B^2 \]
\[ z = z(A, B) = -4A^2 - 8AB + 4B^2 \]
\[ \] Substituting (8) in (2), the corresponding non-zero distinct integral solutions of (1) are given by
\[ x(A, B) = x = -8A^2 - 8AB \\
\]
\[ y(A, B) = y = 8B^2 - 8AB \\
\]
\[ z(A, B) = z = -4A^2 - 4B^2 \\
\]
\[ w(A, B) = w = B^2 + 2AB - A^2 \\
\]
\[ t(A, B) = t = -4A^2 - 8AB + 4B^2 \]
PATTERN II

The equation \( (3) \) can be written as
\[
2w^2 - 16t^2 = z^2 + 1 \quad (9)
\]
Then consider
\[
z = 8a^2 - b^2 \quad (10)
\]
Also
\[
1 = \frac{(n\sqrt{2} + n)(n\sqrt{2} - n)}{n^2} \quad (11)
\]
Where \( n = 1, 2, 3 \ldots \)

Substituting (11) and (10) into (9) reduces to
\[
2w^2 - 16t^2 = (8a^2 - b^2)^2 + 1
\]

Then
\[
\left(\sqrt{2}w + 4t\right) \left(\sqrt{2}w - 4t\right) = \frac{1}{n^2}(2\sqrt{2}a + b)^2 (\sqrt{2}a - b)^2 (n\sqrt{2} + n)(n\sqrt{2} - n)
\]

Now define
\[
\left(\sqrt{2}w + 4t\right) = \frac{1}{n}(8a^2 + b^2 + 4\sqrt{2ab})(n\sqrt{2} + n)
\]
\[
= (8a^2 + b^2 + 8ab) + \sqrt{2}(8a^2 + b^2 + 4ab)
\]

Equating rational and irrational parts, the coefficient values are
\[
w = w(a, b) = 8a^2 + b^2 + 4ab
\]
\[
t = t(a, b) = \frac{1}{4}(8a^2 + b^2 + 8ab)
\]
\[
z = z(a, b) = 8a^2 - b^2
\]

As our interest is on finding integer solutions, it is seen that the values of \( x, y \) and \( z \) are integers when both \( a \) and \( b \) are of the same parity. Thus by taking \( a = 2A, b = 2B \) in (12) and substituting the corresponding values of \( u, v \) in (2) the non-zero integral solutions of (1) are given by
\[
x = x(A, B) = 64A^2 + 16AB
\]
\[
y = y(A, B) = 88B^2 + 16AB
\]
\[
z = z(A, B) = 32A^2 - 4B^2
\]
\[
w = w(A, B) = 32A^2 + 4B^2 + 16AB
\]
\[
t = t(A, B) = 8A^2 + B^2 + 8AB
\]

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES

Journal Articles


Reference Books

