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Integral solutions of Quadratic Diophantine equation $4w^2 - x^2 - y^2 + z^2 = 16t^2$ with five unknowns

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Abstract- The Quadratic Diophantine equation given by $4w^2 - x^2 - y^2 + z^2 = \mathbf{16}t^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Quadratic, integral solutions, polygonal numbers.

INTRODUCTION

The theory of Diophantine equation offers a rich variety of fascinating problems. There are Diophantine problems, which involve quadratic equations with five variables. Quadratic Diophantine equations with five unknowns are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [4-15]. In this communication, we consider yet another interesting quadratic equation $4w^2 - x^2 - y^2 + z^2 = \mathbf{16}t^2$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special Polygonal numbers are presented.

OTATIONS USED

- $t_{m,n}$- Polygonal number of rank 'n' with size 'm'
- CP_n^6 - Centered hexagonal Pyramidal number of rank 'n'
- Gno_n - Gnomic number of rank 'n'
- FN_A^4 - Figurative number of rank 'n' with size 'm'
- Pr_n - Pronic number of rank 'n'
- P_n^m- Pyramidal number of rank 'n' with size 'm'
- SO_n - Stella octagonal number of rank 'n'
- s_n - Star number of rank 'n'
- j_n - Jacobsthal –Lucas number of rank 'n'
- T_n -Triangular number of rank 'n'
- Hex_n- Hexagonal number of rank 'n'
- Obl_n - Oblong number of rank 'n'
- OH_n- Octagonal n umber of rank 'n'
- PT_A – Pentagonal number of rank 'n'

METHOD OF ANALYSIS

The Quadratic Diophantine equation with five unknowns to be solved for its non zero distinct integral solutions is

$$4w^2 - x^2 - y^2 + z^2 = 16t^2 \quad (1)$$

On substituting the linear transformation

$$\left.\begin{aligned} x &= w + z \\ y &= w - z \end{aligned}\right\} \quad (2)$$

in (1), leads to

$$2w^2 - (4t)^2 = z^2 \quad (3)$$

We obtain different patterns of integral solutions to (1) through solving (3) which are illustrated as follows

PATTERN I

Equation (3) can be written as

$$w^2 - (4t)^2 = z^2 - w^2$$

$$(w + 4t)(w - 4t) = (z + w)(z - w) \quad (4)$$

Choice 1:

Equation (4) can be written in the form of ratio as

$$\frac{w + 4t}{z + w} = \frac{z - w}{w - 4t} = \frac{A}{B}, \qquad B \neq 0$$

Which is equivalent to the system of equations

$$(B - A)w + 4Bt - Az = 0$$

$$(A + B)w - 4At - Bz = 0$$

Applying the method of cross multiplication and simplifying, the values of w, t and z are as follows

$$\left.\begin{aligned} w &= w(A,B) = 4A^2 + 4B^2 \\ t &= t(A,B) = A^2 + 2AB - B^2 \\ z &= z(A,B) = 4B^2 + 8AB - 4A^2 \end{aligned}\right\} \quad (5)$$

Substituting (5) in (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x(A,B) &= x = 8B^2 + 8AB \\ y(A,B) &= y = 8A^2 - 8AB \\ z(A,B) &= z = 4B^2 + 8AB - 4A^2 \\ w(A,B) &= w = 4A^2 + 4B^2 \\ t(A,B) &= t = A^2 + 2AB - B^2 \end{aligned}\right\}$$

Properties

1. $z(1,B) + w(1,B) - 16t_B \equiv 0$
2. $z(1,B(B+2)) + w(1,B(B+2)) - 48P_A^3 \equiv 0$
3. $x(A,A) + y(A,A)$ can be expressed as a perfect square
4. $x(1,1)$ is a perfect square
5. $t(2,1)$ as a carol number
6. $x(0,1) - t(1,2)$ is a Kynea number
7. $w(1,1) + t(0,1)$ is a Woodall number
8. The following expression represents a Nasty number
   a. $z(1,2) - w(0,1)$
   b. $x(2,1)$
   c. $w(1,2) + z(0,1)$
9. The following expression represents a perfect number
   a. $x(1,2) - w(1,2)$
   b. $z(1,2)$

Choice 2:

Equation (4) can be rewritten in the form of ratio as

$$\frac{w+4t}{z-w} = \frac{z+w}{w-4t} = \frac{A}{B}\ , \qquad B \neq 0$$

Which is equivalent to the system of equations

$$(A+B)w + 4Bt - Az = 0$$
$$(B-A)w + 4At + Bz = 0$$

Applying the method of cross multiplication and simplifying, the values of w, t and z are as follows

$$\left.\begin{aligned} w &= w(A,B) = 4A^2 + 4B^2 \\ t &= t(A,B) = A^2 - 2AB - B^2 \\ z &= z(A,B) = 4A^2 + 8AB - 4B^2 \end{aligned}\right\} \quad (6)$$

Substituting (6) in (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x(A,B) &= x = 8A^2 + 8AB \\ y(A,B) &= y = 8B^2 - 8AB \\ z(A,B) &= z = 4A^2 + 4B^2 \\ w(A,B) &= w = A^2 - B^2 - 2AB \\ t(A,B) &= t = 4A^2 + 8AB - 4B^2 \end{aligned}\right\}$$

Choice 3:

Equation (4) can be rewritten in the form of ratio as

$$\frac{w-4t}{z+w} = \frac{z-w}{w+4t} = \frac{A}{B}\ , \qquad B \neq 0$$

Which is equivalent to the system of equations

$$(B-A)w - 4Bt - Az = 0$$
$$(-B-A)w - 4At + Bz = 0$$

Applying the method of cross multiplication and simplifying, the values of w, t and z are as follows

$$\left.\begin{aligned} w &= w(A,B) = -4A^2 - 4B^2 \\ t &= t(A,B) = A^2 + 2AB - B^2 \\ z &= z(A,B) = 4A^2 - 8AB - 4B^2 \end{aligned}\right\} \quad (7)$$

Substituting (7) in (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x(A,B) &= x = -8B^2 - 8AB \\ y(A,B) &= y = -8A^2 + 8AB \\ z(A,B) &= z = 4A^2 - 8AB - 4B^2 \\ w(A,B) &= w = -4A^2 - 4B^2 \\ t(A,B) &= t = A^2 + 2AB - B^2 \end{aligned}\right\}$$

Choice 4:

Equation (4) can be rewritten in the form of ratio as

$$\frac{w-4t}{z-w} = \frac{z+w}{w+4t} = \frac{A}{B}\ , \qquad B \neq 0$$

Which is equivalent to the system of equations

$$(B+A)w - 4Bt - Az = 0$$
$$(B-A)w - 4At + Bz = 0$$

Applying the method of cross multiplication and simplifying, the values of w, t and z are as follows

$$\left.\begin{aligned} w &= w(A,B) = -4A^2 - 4B^2 \\ t &= t(A,B) = -A^2 + 2AB + B^2 \\ z &= z(A,B) = -4A^2 - 8AB + 4B^2 \end{aligned}\right\} \quad (8)$$

Substituting (8) in (2), the corresponding non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x(A,B) &= x = -8A^2 - 8AB \\ y(A,B) &= y = 8B^2 + 8AB \\ z(A,B) &= z = -4A^2 - 4B^2 \\ w(A,B) &= w = B^2 + 2AB - A^2 \\ t(A,B) &= t = -4A^2 - 8AB + 4B^2 \end{aligned}\right\}$$

PATTERN II

The equation (3) can be written as

$$2w^2 - 16t^2 = z^2 * 1 \quad (9)$$

Then consider

$$z = 8a^2 - b^2 \quad (10)$$

Also

$$1 = \frac{(n\sqrt{2}+n)(n\sqrt{2}-n)}{n^2} \quad (11)$$

Where n=1,2,3…….

Substituting (11) and (10) into (9) reduces to

$$2w^2 - 16t^2 = (8a^2 - b^2)^2 * 1$$

Then

$$(\sqrt{2}w + 4t)(\sqrt{2}w - 4t) = \frac{1}{n^2}\left(2\sqrt{2}a + b\right)^2\left(2\sqrt{2}a - b\right)^2(n\sqrt{2}+n)(n\sqrt{2}-n)$$

Now define

$$(\sqrt{2}w + 4t) = \frac{1}{n}(8a^2 + b^2 + 4\sqrt{2}ab)(n\sqrt{2}+n) = (8a^2 + b^2 + 8ab) + \sqrt{2}(8a^2 + b^2 + 4ab)$$

Equating rational and irrational parts, the coefficient values are

$$\left.\begin{aligned} w &= w(a,b) = 8a^2 + b^2 + 4ab \\ t &= t(a,b) = \frac{1}{4}(8a^2 + b^2 + 8ab) \\ z &= z(a,b) = 8a^2 - b^2 \end{aligned}\right\} \quad (12)$$

As our interest is on finding integer solutions, it is seen that the values of x, y and z are integers when both a and b are of the same parity. Thus by taking $a = 2A$, $b = 2B$ in (12) and substituting the corresponding values of u, v in (2) the non-zero integral solutions of (1) are given by

$$\left.\begin{aligned} x &= x(A,B) = 64A^2 + 16AB \\ y &= y(A,B) = 8B^2 + 16AB \\ z &= z(A,B) = 32A^2 - 4B^2 \\ w &= w(A,B) = 32A^2 + 4B^2 + 16AB \\ t &= t(A,B) = 8A^2 + B^2 + 8AB \end{aligned}\right\}$$

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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