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The Common Neighbor Polynomial of Some Graph Constructions

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Abstract- Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered vertex pair of distinct vertices of G. The i-common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v$ and $|N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n-2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the common neighbor polynomial of G [3]. In this paper we study common neighbor polynomial of some graph constructions.

Key Words: Common neighbor set, Common neighbor polynomial

1 Introduction

Let $G(V, E)$ be a simple graph of order n with vertex set V and edge set E. Let (u, v) denotes an unordered pair of distinct vertices of G. The i-common neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v$ and $|N(u) \cap N(v)| = i\}$, for $0 \leq i \leq n-2$. The polynomial $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$ is defined as the common neighbor polynomial of G [3]. In [3] the present authors derived the common neighbor polynomial of some well known graphs. The common neighbor polynomial of some graph operations were discussed in [4].

Common neighbor polynomial may be useful in the study of social networks, citation networks etc. "While modelling the structure of a social network system, usually pairs of individuals with shared interests are represented by pairs of vertices with common neighbors. The number of such common neighbors serves as a measure of consensus and proclivities between the corresponding pair of individuals"[5].

In this paper we study common neighbor polynomial of some graphs and graph constructions.

2 Main results

Let v_0 be a specific vertex of a graph G. Let $G_{v_0}(m)$ be a graph obtained from G by identifying the vertex V_0 of G with an end vertex of the path P_{m+1} with $m+1$ vertices [6].

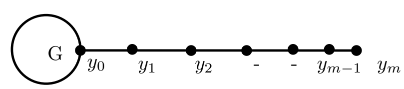

Figure 1: The graph $G_{v_0}(m)$

Theorem 1. *Let G be a graph with n vertices and let $v_0 \in V(G)$. If $deg(v_0) = d$, we have $N[G_{v_0}(m); x] = N[G; x] + (m + d - 1)x + mn - d + \binom{m-1}{2}$.*

Proof. Let $y_0, y_1, \ldots, y_m$ be the vertices of the path P_{m+1}. Let the vertex v_0 of G be identified with the end vertex y_0 of P_{m+1}. Let (u, v) be any pair of vertices of $G_{v_0}(m)$. We consider 3 cases:
Case(i) Let $u, v \in V(G)$.
Then the number of vertex pairs (u, v) with i common neighbors in $G_{v_0}(m)$ equals $|N(G, i)|$.

Case(ii) Let $u, v \in \{y_1, y_2, \ldots, y_m\}$.
Then the number of vertex pairs (u, v) with i common neighbors in $G_{v_0}(m)$ equals $|N(P_m, i)|$.
Case(iii) Let $v \in \{y_1, y_2, \ldots, y_m\}$ and $u \in V(G)$.
If $u = y_0$, then (u, y_2) has one common neighbor and if u is a neighbor of y_0, then (u, y_1) has one common neighbor. Thus $d+1$ pairs of vertices under this case have 1 common neighbor. All other $(mn - d - 1)$ vertices under this case have no common neighbors.

It follows that

$$\begin{aligned} N[G_{v_0}(m); x] &= N[G; x] + N[P_m; x] + (d+1)x \\ &\qquad +(mn - d - 1) \\ &= N[G; x] + (m-2)x + \binom{m-1}{2} \\ &\qquad +1 + (d+1)x + (mn - d - 1) \\ &= N[G; x] + (m + d - 1)x + mn - d + \binom{m-1}{2}. \end{aligned}$$

This completes the proof. □

Let a and b be two specific vertices of a graph G. Let $G'_{a,b}(m)$ or simply, $G'(m)$ be a graph obtained from G by identifying the vertices a and b of G with the two end vertices of a path P_m [6].

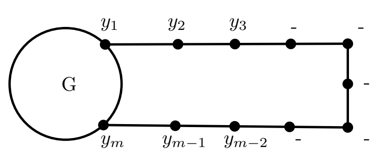

Figure 2: The graph $G'(m)$

Theorem 2. *Let G be a graph with n vertices. Let a, b be two specific vertices of G. Then for $m > 2$, we have $N[G'(m); x] = N[G; x] + (m + d - 2)x + \binom{m-3}{2} + n(m-2) - (d+1)$ where d denotes the sum of degrees of the vertices a and b in G.*

Proof. Let $y_1, y_2, \ldots, y_m$ be the vertices of a path P_m. Let the vertices a, b of G be identified with the end vertices y_1 and y_m of P_m respectively. Let (u, v) be any pair of vertices of $G'(m)$.

Here we consider the following 3 cases:
Case(i) Let $u, v \in V(G)$.
Then the number of vertex pairs (u, v) with i common neighbors in $G'(m)$ equals $|N(G, i)|$.
Case(ii) Let $u, v \in \{y_2, y_3, \ldots, y_{m-1}\}$.
Then the number of vertex pairs (u, v) with i common neighbors in $G'(m)$ equals $|N(P_{m-2}, i)|$.
Case(iii) Let $v \in \{y_2, y_3, \ldots, y_{m-1}\}$ and $u \in V(G)$.
If $u = y_1$, then (u, y_3) has one common neighbor and if $u = y_m$, then (u, y_{m-2}) has one common neighbor. If $uy_1 \in E(G)$ then (u, y_2) has one common neighbor in $G'(m)$ and if $uy_m \in E(G)$ then (u, y_{m-1}) has one common neighbor in $G'(m)$. Thus $d+2$ pairs of vertices (u, v) have 1 common neighbor in $G'(m)$. All other $n(m-2) - (d+2)$ vertex pairs under this case have no common neighbors.
It follows that

$$\begin{aligned} N[G'(m); x] &= N[G; x] + N[P_{m-2}; x] \\ &\quad + (d+2)x + n(m-2) - (d+2) \\ &= N[G; x] + (m-4)x + \binom{m-3}{2} \\ &\quad + 1 + (d+2)x + n(m-2) - (d+2) \\ &= N[G; x] + (m + d - 2)x \\ &\quad + \binom{m-3}{2} + n(m-2) - (d+1). \end{aligned}$$

This completes the proof. □

Let G_1 and G_2 be two disjoint graphs. Let $(G_1, G_2)_{u,v}(m)$ be a graph obtained by identifying the vertices u of G_1 and v of G_2 with the end vertices y_1 and y_m respectively, of a path P_m.

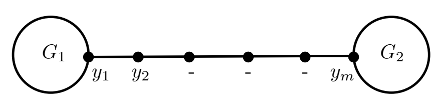

Figure 3: The graph $(G_1, G_2)_{u,v}(m)$

Theorem 3. *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices respectively. Let $u \in V(G_1)$ is of degree d_1 and $v \in V(G_2)$ is of degree d_2. Then $N[(G_1, G_2)_{u,v}(m); x] = N[G_1; x] + N[G_2; x] +$*

$N[P_{m-2};x]+(d_1+d_2+2)x+(n_1+n_2)(m-2)-(d_1+d_2)+n_1n_2-2$ *where* $m>3$.

Proof. Let $y_1,y_2,\ldots,y_m$ be the vertices of the path P_m. Let the vertex u of G_1 be identified with the end vertex y_1 of P_m and let the vertex v of G_2 be identified with the vertex y_m. Let (x,y) be any pair of vertices of $(G_1,G_2)_{u,v}(m)$. We consider 6 cases:
Case(i) Let $x,y\in V(G_1)$.
Then the number of vertex pairs (x,y) with i common neighbors in $(G_1,G_2)_{u,v}(m)$ equals $|N(G_1,i)|$.
Case(ii) Let $x,y\in V(G_2)$.
Then the number of vertex pairs (x,y) with i common neighbors in $(G_1,G_2)_{u,v}(m)$ equals $|N(G_2,i)|$.
Case(iii) Let $y\in\{y_2,y_3,\ldots,y_{m-1}\}$ and $x\in V(G_1)$.
In this case, if $x=u$, the vertex pair (x,y_3) has exactly one common neighbor y_2 and if x is a neighbor of u in G_1, then there are d_1 pairs of vertices of the form (x,y_2) which have exactly one common neighbor y_1. The remaining $n_1(m-2)-(1+d_1)$ vertex pairs have no common neighbors.
Case(iv) Let $y\in\{y_2,y_3,\ldots,y_{m-1}\}$ and $x\in V(G_2)$.
As in Case(iii), the vertex pair (y_m,y_{m-2}) has exactly one common neighbor y_{m-1} and d_2 pairs of vertices has exactly one common neighbor y_m. The remaining $n_2(m-2)-(1+d_2)$ vertex pairs have no common neighbors.
Case(v) Let $x,y\in\{y_2,y_3,\ldots,y_{m-2}\}$.
Then the number of pairs of vertices having i common neighbors equals $|N(P_{m-2},i)|$.
Case(vi) Let $x\in V(G_1)$ and $y\in V(G_2)$.
Since $m>3$, all the n_1n_2 pairs of vertices (x,y) under this case have no common neighbors.
Thus it follows that

$$\begin{aligned}N[(G_1,G_2)(m);x] &= N[G_1;x]+N[G_2;x]\\ &+N[P_{m-2};x]+(1+d_1)x+n_1(m-2)\\ &-(d_1+1)+(1+d_2)x+n_2(m-2)\\ &-(d_2+1)+n_1n_2\\ &= N[G_1;x]+N[G_2;x]+N[P_{m-2};x]\\ &+(d_1+d_2+2)x+(n_1+n_2)(m-2)\\ &-(d_1+d_2)+n_1n_2-2.\end{aligned}$$

□

A flower graph $f_{n\times m}$ is a graph with a n-cycle and n number of m-cycles each intersects with the n-cycle on a unique single edge [1].

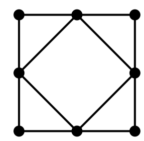

Figure 4: The flower graph $f_{4\times 3}$

Theorem 4. *If $f_{n\times m}$ is a flower graph, then, the following results hold:*

1. *If* $m\neq 4$, $N[f_{n\times m};x]=N[C_n;x]+n\,N[P_{m-2};x]+5nx+(m-2)n^2+\binom{n}{2}(m-2)^2-5n$.
2. *If* $m=4$, $N[f_{n\times m};x]=N[C_n;x]+2nx^2+3nx+4n^2-6n$.

Proof. Let C_n be the inner cycle and $C_m^1,C_m^2,\ldots,C_m^n$ be the m-cycles having one of the edges common to C_n. Let $v_1,v_2,\ldots,v_n$ be the vertices of C_n and for each $j\in\{1,2,\ldots,n\}$, let $U_j=\{u_1^j,u_2^j,\ldots,u_{m-2}^j\}$ be the set of $m-2$ vertices which form the m-cycle C_m^j together with the edge v_jv_{j+1} of C_n. Let (u,v) be any pair of vertices of $f_{n,m}$. We consider 3 cases.
Case(i) Let $u,v\in V(C_n)$.
Then the number of pairs (u,v) with i common neighbors in $f_{n,m}$ equals $|N(C_n,i)|$.
Case(ii) Let $u,v\in U_j$ where $j\in\{1,2,\ldots,n\}$.
Then for each $j\in\{1,2,\ldots,n\}$ the number of pairs (u,v) with i common neighbors in $f_{n,m}$ equals $|N(P_{m-2},i)|$.
Case(iii) Let $u\in U_j$ and $v\in U_k$ where $j,k\in\{1,2,\ldots,n\}$ and $j\neq k$. Then the n pairs (u_{m-2}^{j-1},u_1^j) has exactly one common neighbor v_j where the index j is taken modulo m. All other $\binom{n}{2}(m-2)^2-n$ pairs of vertices under this case have no common neighbors.
Case(iv) Let $u\in V(C_n)$ and $v\in U_j$ where $j\in\{1,2,\ldots,n\}$.
Then the pairs of the form (u_1^j,v_{j-1}) and (u_{m-2}^{j-1},v_{j+1}) has exactly one common neighbor v_j

. Also the pairs (u_1^j, v_{j+1}) has exactly one common neighbor v_j if $m \neq 4$ and has two common neighbors u_{m-2}^j and v_j if $m = 4$. Similarly, the pairs (u_{m-2}^j, v_j) has one common neighbor v_{j+1} if $m \neq 4$ and has two common neighbors u_1^j and v_{j+1} if $m = 4$. All other $(m-2)n^2 - 4n$ pairs of vertices under this case have no common neighbors.
It follows that

1. If $m \neq 4$, $N[f_{n\times m}; x] = N[C_n; x] + n\ N[P_{m-2}; x] + 4nx + (m-2)n^2 - 4n + nx + \binom{n}{2}(m-2)^2 - n$.
   $= N[C_n; x] + n\ N[P_{m-2}; x] + 5nx + (m-2)n^2 + \binom{n}{2}(m-2)^2 - 5n$.
2. If $m = 4$, $N[f_{n\times m}; x] = N[C_n; x] + n\ N[P_2; x] + 2nx^2 + 2nx + 2n^2 - 4n + nx + 4\binom{n}{2} - n$.
   $= N[C_n; x] + 2nx^2 + 3nx + 4n^2 - 6n$.

This completes the proof. □

A chaplet graph[7] $C_p \odot C_q^t$ where $p, q, t \geq 3$ is obtained by taking one point union of t-copies of the cycle C_q and attaching the same to each vertex of the cycle C_p.

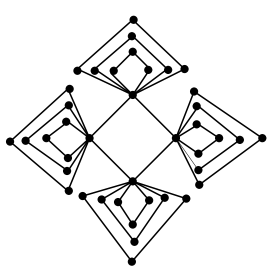

Figure 5: The chaplet graph $C_4 \odot C_4^3$

Theorem 5. $N[C_p \odot C_q^t; x] = N[C_p; x] + tpN[C_q; x] + [4tp + 3pt(t-1)]x + \binom{p}{2}t^2(q-1)^2 + p(q^2-2q-5)\binom{t}{2} + [(p-1)(q-1)-4]tp$.

Proof. Let $u_1, u_2, \ldots, u_p$ be the vertices of the cycle C_p. For $j \in \{1, 2, \ldots, t\}$ and $k \in \{1, 2, \ldots, p\}$, let $u_k, u_{k1}^j, u_{k2}^j, \ldots, u_{k(q-1)}^j$ be the vertices of j^{th} copy of the cycle C_q attached to the vertex u_k of C_p. Let (u, v) be any pair of vertices of $C_p \odot C_q^t$. We consider the following cases:
Case(i) Let $u, v \in \{u_1, u_2, \ldots, u_p\}$.
In this case, the number of vertex pairs (u, v) with i common neighbors equals $|N(C_p, i)|$.
Case(ii) Let $u, v \in \{u_k, u_{k1}^j, u_{k2}^j, \ldots, u_{k(q-1)}^j\}$ where $j \in \{1, 2, \ldots, t\}$ and $k \in \{1, 2, \ldots, p\}$.
Fixing the variables j and k, the number of vertex pairs (u, v) with i common neighbors equals $|N(C_q, i)|$ and there are tp choices for fixing j and k.
Case(iii) Let $u \in \{u_{k1}^j, u_{k2}^j, \ldots, u_{k(q-1)}^j\}$ and $v \in \{u_1, u_2, \ldots, u_{k-1}, u_{k+1}, \ldots, u_p\}$ where $j \in \{1, 2, \ldots, t\}$ and $k \in \{1, 2, \ldots, p\}$.
In this case, pairs of vertices of the form (u_{k1}^j, u_{k+1}), (u_{k1}^j, u_{k-1}), $(u_{k(q-1)}^j, u_{k+1})$ and $(u_{k(q-1)}^j, u_{k-1})$ have exactly one common neighbor u_k and there are $4tp$ pairs of vertices of this form. All other vertices under this case have no common neighbors and there are $(p-1)(q-1)tp - 4tp$ such pairs.
Case(iv) Let $u \in \{u_{k1}^j, u_{k2}^j, \ldots, u_{k(q-1)}^j\}$, $v \in \{u_{k1}^l, u_{k2}^l, \ldots, u_{k(q-1)}^l\}$ where $j, l \in \{1, 2, \ldots, t\}$, $k \in \{1, 2, \ldots, p\}$ and $j \neq l$.
In this case, pairs of vertices of the form (u_{k1}^j, u_{k1}^l),$(u_{k(q-1)}^j, u_{k(q-1)}^l)$ and $(u_{k1}^j, u_{k(q-1)}^l)$ have exactly one common neighbor u_k and there are $2p\binom{t}{2} + pt(t-1) = 4p\binom{t}{2}$ pairs of vertices of this form. All the remaining vertices under this case have no common neighbors and the number of such vertices are given by $\binom{t}{2}p(q-1)^2 - 4p\binom{t}{2}$ which equals $p(q^2-2q-3)\binom{t}{2}$.
Case(v) Let $u \in \{u_{k1}^j, u_{k2}^j, \ldots, u_{k(q-1)}^j\}$, $v \in \{u_{s1}^l, u_{s2}^l, \ldots, u_{s(q-1)}^l\}$ where $j, l \in \{1, 2, \ldots, t\}$ and $k, s \in \{1, 2, \ldots, p\}$ and $k \neq s$.
In this case the pairs of vertices (u, v) have no common neighbors and there are $\binom{p}{2}t^2(q-1)^2$ such vertex pairs. Hence it follows that
$N[C_p \odot C_q^t; x] = N[C_p; x] + tpN[C_q; x] + 4tp\ x$
$+ [(p-1)(q-1)-4]tp + 4p\binom{t}{2}x$
$+ p(q^2-2q-3)\binom{t}{2} + \binom{p}{2}t^2(q-1)^2$
$= N[C_p; x] + tpN[C_q; x] + [4tp + 2pt(t-1)]x$
$+ \binom{p}{2}t^2(q-1)^2 + p(q^2-2q-3)\binom{t}{2}$
$+ [(p-1)(q-1)-4]tp$. This completes the proof. □

A snake graph $S_{n,m}$ is obtained from a path graph P_n replacing each edge of P_n by the cycle graph C_m [2]. $S_{n,3}$ is known as the triangular snake graph and $S_{n,4}$ the rectangular snake graph.

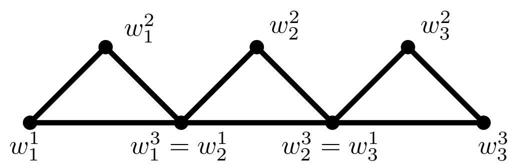

Figure 6: The snake graph $S_{3,3}$

Theorem 6. *For a snake graph $S_{n,m}$, $N[S_{n,m};x] = nN[C_m;x] + 4(n-1)x + [(m-1)^2 - 4](n-1) + (m-1)^2\binom{n-1}{2}$.*

Proof. Let the vertices of the i^{th} cycle of $S_{n,m}$ be represented by $w_i^1, w_i^2, \ldots, w_i^m$ respectively. Let (u,v) be any pair of vertices of $S_{n,m}$. We will consider 3 cases:
Case(i) Let $u, v \in \{w_i^1, w_i^2, \ldots, w_i^m\}$; $i \in \{1, 2, \ldots, n\}$.
Then for each i, the number of vertex pairs (u,v) with k common neighbors equals $|N(C_m,k)|$.
Case(ii) Let $u \in \{w_i^1, w_i^2, \ldots, w_i^{m-1}\}$ and $v \in \{w_{i+1}^2, w_{i+1}^3, \ldots, w_{i+1}^m\}$; $i \in \{1, 2, \ldots, n-1\}$.
Then the pairs (w_i^1, w_{i+1}^2), (w_i^1, w_{i+1}^m), (w_i^{m-1}, w_{i+1}^2), (w_i^{m-1}, w_{i+1}^m) have exactly one common neighbor w_i^m and there are $4(n-1)$ such pairs. The remaining $[(m-1)^2 - 4](n-1)$ pairs under this case have no common neighbors.
Case(iii) Let $u \in \{w_i^1, w_i^2, \ldots, w_i^{m-1}\}$ and $v \in \{w_j^2, w_j^3, \ldots, w_j^m\}$; $i \in \{1, 2, \ldots, n-2\}$ and $j \in \{i+2, i+3, \ldots, n\}$.
The vertex pairs under this case have no common neighbors and there are $(m-1)^2\sum_{i=1}^{n-2}(n-i-1) = (m-1)^2\binom{n-1}{2}$ such pairs.
It follows that
$N[S_{n,m};x] = nN[C_m;x] + 4(n-1)x + [(m-1)^2 - 4](n-1) + (m-1)^2\binom{n-1}{2}$. ☐

Corollary 7. *For a triangular snake graph $S_{n,3}$, we have the following: $N[S_{n,3};x] = nN[C_3;x] + 4(n-1)x + 4\binom{n-1}{2}$.*

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