International Journal of Research in Advent Technology, Vol.6, No.9, September 2018
E-ISSN: 2321-9637
Available online at www.ijrat.org

On Monoid Recognizable *l*-Fuzzy Languages

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Abstract-Here we show that the class of monoid recognizable *l*-fuzzy languages is closed under Boolean operations. Also we prove that the syntactic monoid of a recognizable *l*-fuzzy language is finite and every finite monoid is a syntactic monoid of a recognizable *l*-fuzzy language.

Index Terms: *l*-fuzzy languages; Syntactic congruence; Syntactic monoid.

1. INTRODUCTION

Zadeh [12] introduced the notion of a fuzzy subset of an ordinary set as a method of representing uncertainty. Later it came as a useful tool for describing real-life problems. Zadeh and Lee [6] generalized the classical notion of languages to the concept of fuzzy languages in 1969. A detailed account of the latest developments in the theory of automata and fuzzy languages was given in [7]. In [8] Petkovic introduced the notion of syntactic monoid of a fuzzy language and proved that every finite monoid is the syntactic monoid of a recognizable fuzzy language.

In this paper we discussed monoid recognizability of *l*-fuzzy languages. We introduce the concept of syntactic monoid of a *l*-fuzzy language and studied its properties. Also we prove that every finite monoid is a syntactic monoid of a recognizable *l*-fuzzy language.

2. PRELIMINARIES

In this section we recall the basic definitions, results and notations that will be used in the sequel. All undefined terms are as in [7, 9]. A lattice is a partially ordered set in which every subset $\{a, b\}$ consisting of two element has a least upper bound ($a \vee b$) and a greatest lower bound ($a \wedge b$). A lattice l is said to be bounded if it has a greatest element 1 and a least element 0. A lattice l is said to be distributive if for any element a, b and c of l, we have the following distributive properties.

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Let l be a bounded lattice with greatest element 1 and least element 0 and let $a \in l$. An element $b \in l$ is called complement of a if $a \vee b = 1$ and $a \wedge b = 0$. Complements need not be unique. But if l is a bounded distributive lattice then complements are unique if they exist (cf. [10]). A lattice l is called complemented if it is bounded and if every element in l has a complement. A lattice l is called a complete lattice if every nonempty subset of l has greatest lower bound and least upper bound in l. Thus every finite lattice is complete.

A semigroup consists of a nonempty set M on which an associative binary operation $\cdot$ is defined and is denoted by $(M, \cdot)$. If there exists an element 1 satisfying $m \cdot 1 = m = 1 \cdot m$ for all $m \in M$, then M is called a monoid (semigroup with identity). Let $(M, \cdot)$ be a monoid, then a nonempty subset M_1 of M is called a submonoid of M if it is closed with respect to the induced binary operation.

Let A be a nonempty finite set called an alphabet. Elements of A are called letters. A finite sequence of letters of A is called a word. The length of the word w, in symbols $|w|$, is the number of letters of A occurring in w. A word of length zero is called empty word and is denoted by ε. A^+ denotes the set of all nonempty words over an alphabet A and $A^* = A^+ \cup \{\varepsilon\}$ is a monoid under the operation concatenation, called free monoid over A. A subset of A^* is called the language L over an alphabet A.

Let $L \subseteq A^*$. Then L is recognizable if there exists a finite monoid M and a homomorphism $\phi: A^* \to M$ such that $L = \phi^{-1}(P)$, where $P \subseteq M$. Also we say that M recognizes L.

Let $L \subseteq A^*$. For $u, v \in A^*$, we define a relation P_L by

$$uP_L v \text{ if } xuy \in L \Leftrightarrow xvy \in L,$$

for all $x, y \in A^*$. Then P_L is a congruence, called the syntactic congruence. The quotient monoid $A^* / P_L = M(L)$ is called the syntactic monoid and the canonical homomorphism $\eta_L: \underline{A^* \to M(L)}$ is called the syntactic morphism of L.

3. l-FUZZY LANGUAGES

Let l be a complete complemented distributive lattice.
Any function λ from A^* into l is called a l-fuzzylanguage over the alphabet A.

Example3.1. Let $l = (\{\{c\},\{d\},\{c,d\},\emptyset\},\cup,\cap)$ and let $A = \{a,b\}$ be a complete complemented distributive lattice on the set $\{c,d\}$. The function $\lambda : A^* \to l$ defined by

$$\lambda(u) = \begin{cases} \{c\} & \text{if } u \in aA^* \\ \{d\} & \text{if } u \in bA^* \end{cases}$$

is a l-fuzzy language over A.

Definition 3.2. *Let λ be a l-fuzzy language over an alphabet A. Then λ is recognizable if there exist a finite monoid M, a homomorphism $\varphi : A^* \to M$ and a l-fuzzy subset $\pi : M \to l$ such that $\lambda = \pi\varphi^{-1}$ where $\pi\varphi^{-1}(u) = \pi(\varphi(u))$, $u \in A^*$. We also say that the monoid M recognizes λ by a morphism φ.*

Example 3.3. *χ_{A^*} is a recognizable l-fuzzy language.*

Now we define the complement $\overline{\lambda}$ of a l-fuzzy language λ as

$$\bar{\lambda}(u) = \overline{\lambda(u)}$$

where $\overline{\lambda(u)}$ denotes the complement of $\lambda(u)$ in l.

For l-fuzzy languages λ_1, λ_2 over A, their join($\vee$) and meet ($\wedge$) are defined by

$$(\lambda_1 \vee \lambda_2)(u) = \lambda_1(u) \vee \lambda_2(u)$$

and

$$(\lambda_1 \wedge \lambda_2)(u) = \lambda_1(u) \wedge \lambda_2(u).$$

Theorem 3.4. *Let λ, λ_1, λ_2 be recognizable l-fuzzy languages over an alphabet A. Then we have the following*

(1) $\lambda_1 \vee \lambda_2$ is recognizable.
(2) $\lambda_1 \wedge \lambda_2$ is recognizable.
(3) $\bar{\lambda}$ is recognizable.

Proof. **(1)** Since λ_1 and λ_2 are recognizable, there exist finite monoids M_1 and M_2, homomorphisms $\varphi_1 : A^* \to M_1$ and $\varphi_2 : A^* \to M_2$ and l-fuzzy subsets $\pi_1 : M_1 \to l$ and $\pi_2 : M_2 \to l$ such that $\lambda_1 = \pi_1\varphi_1^{-1}$ and $\lambda_2 = \pi_2\varphi_2^{-1}$. Define a map $\theta : A^* \to M_1 \times M_2$ by

$$\theta(u) = (\varphi_1(u), \varphi_2(u)).$$

For $u_1, u_2 \in A^*$, We have

$$\begin{aligned} \theta(u_1u_2) &= (\varphi_1(u_1u_2), \varphi_2(u_1u_2)) \\ &= (\varphi_1(u_1)\varphi_1(u_2), \varphi_2(u_1)\varphi_2(u_2)) \\ &= (\varphi_1(u_1),\varphi_2(u_1))(\varphi_1(u_2),\varphi_2(u_2)) \\ &= \theta(u_1)\theta(u_2). \end{aligned}$$

So θ is a homomorphism. Define $\pi : M_1 \times M_2 \to l$ by

$$\pi(m_1,m_2) = \pi_1(m_1) \vee \pi_2(m_2).$$

Since π is well defined, π is a l-fuzzy subset of $M_1 \times M_2$.
For $u \in A^*$, we have

$$\begin{aligned} \pi\theta^{-1}(u) &= \pi(\theta(u)) = \pi((\varphi_1(u),\varphi_2(u))) \\ &= \pi_1(\varphi_1(u)) \vee \pi_2(\varphi_2(u)) \\ &= \pi_1\varphi_1^{-1}(u) \vee \pi_2\varphi_2^{-1}(u) \\ &= \lambda_1(u) \vee \lambda_2(u) = (\lambda_1 \vee \lambda_2)(u). \end{aligned}$$

So $\pi\theta^{-1} = \lambda_1 \vee \lambda_2$. Hence $\lambda_1 \vee \lambda_2$ is recognized by $M_1 \times M_2$.

(ii) (2) The map $\phi : M_1 \times M_2 \to l$ defined by

$$\phi(m_1,m_2) = \pi_1(m_1) \wedge \pi_2(m_2)$$

is well defined. So ϕ is a l-fuzzy subset of $M_1 \times M_2$. Thus

$$\begin{aligned} \phi\theta^{-1}(u) &= \phi(\theta(u)) \\ &= \phi((\varphi_1(u),\varphi_2(u))) \\ &= \pi_1(\varphi_1(u)) \wedge \pi_2(\varphi_2(u)) \\ &= \pi_1\varphi_1^{-1}(u) \wedge \pi_2\varphi_2^{-1}(u) \\ &= \lambda_1(u) \wedge \lambda_2(u) = (\lambda_1 \wedge \lambda_2)(u), \end{aligned}$$

for all $u \in A^*$. Hence $\lambda_1 \wedge \lambda_2 = \phi\theta^{-1}$. Therefore $M_1 \times M_2$ recognizes $\lambda_1 \wedge \lambda_2$.

(3) Since λ is recognizable, there exist a finite monoid M, an onto homomorphism $\varphi : A^* \to M$ and a l-fuzzy subset π on M such that $\lambda = \pi\varphi^{-1}$ where $\lambda(u) = \pi\varphi^{-1})(u) = \pi(\varphi(u))$. Define π_1 from M to l by

$$\pi_1(m) = \overline{\pi(m)}$$

$$\begin{aligned} \text{Then} (\pi_1\varphi^{-1})(u) &= \pi_1(\varphi(u)) \\ &= \overline{\pi(\varphi(u)})) \\ &= \overline{(\pi\varphi^{-1})(u)} \\ &= \overline{\lambda(u)} \\ &= \bar{\lambda}(u) \end{aligned}$$

for all $u \in A^*$. Therefore $\pi_1\varphi^{-1} = \bar{\lambda}$. Thus $\bar{\lambda}$ is a recognizable language.

The class of all recognizable l-fuzzy languages over A is denoted by $lF(A^*)$. By Example 3.3, we have $\chi_{A^*} \in lF(A^*)$. Thus $lF(A^*)$ is a nonempty subclass of the class of all l-fuzzy languages. From Theorem 3.4, it follows that $lF(A^*)$ is closed under join(∨), meet(∧) and complementation. Moreover, we have the following.

Corollary 3.5. *$lF(A^*)$ is a Boolean Algebra.*

The following theorem gives a necessary and sufficient condition for the recognizability of l-fuzzy languages.

Theorem 3.6. *Let λ be a l-fuzzy language over an alphabet A. Then a monoid M recognizes λ by a homomorphism $\varphi : A^* \to M$ if and only if $\ker\varphi$ saturates λ.*

Proof. Assume that the monoid M recognizes λ by a homomorphism $\varphi : A^* \to M$. Then there exists a l-fuzzy subset π of M such that $\lambda = \pi\varphi^{-1}$ where $\lambda(u) = (\pi\varphi^{-1})(u) = \pi(\varphi(u))$, $u \in A^*$. Let u and v belongs to A^*. Then $(u,v) \in \ker\varphi$ if and only if $\varphi(u) = \varphi(v)$. Thus $\pi(\varphi(u)) = \pi(\varphi(v))$. That is, $\pi\varphi^{-1}(u) = \pi\varphi^{-1}(v)$. Hence $\lambda(u) = \lambda(v)$. Therefore $\ker\varphi$ saturates λ.

Conversely assume that $\varphi : A^* \to M$ is a homomorphism and $\ker\varphi$ saturates λ.
Define a function $\pi : \varphi(A^*) \to l$ by

$$\pi(\varphi(u)) = \lambda(u), \quad u \in A^*$$

If $\varphi(u) = \varphi(v)$, then $(u,v) \in \ker\varphi$. Since $\ker\varphi$ saturates λ, we have $\lambda(u) = \lambda(v)$. So π is well defined. Let $\pi_1 : M \to l$ be a function such that $\pi_1|_{\varphi(A^*)} = \pi$. Then , for all $u \in A^*$, we have $(\pi_1\varphi)(u) = \pi_1(\varphi(u)) = \pi(\varphi(u)) = \lambda(u)$. So $\lambda = \pi_1\varphi^{-1}$. Thus M recognizes λ.

4. SYNTACTIC CONGRUENCE

Let λ be a l-fuzzy language over A. Define a relation $(\sim_\lambda)$ on A^* as follows:

For $u, v \in A^*$, $u \sim_\lambda v$ if and only if

$$\lambda(puq) = \lambda(pvq), \text{for all } p, q \in A^*.$$

Then the relation $\sim_\lambda$ is a congruence on A^* called syntactic congruence of λ. The quotient monoid $A^*/\sim_\lambda = \mathrm{Syn}(\lambda)$ is called syntactic monoid of λ. The assignment $u \to [u]_{\sim\lambda}$ defines a homomorphism $\eta_\lambda : A^* \to \mathrm{Syn}(\lambda)$ called the syntactic homomorphism of λ.

Let $u, v \in A^*$ and let $(u,v) \in \ker(\eta_\lambda)$. Then $\eta_\lambda(u) = \eta_\lambda(v)$. That is, $[u]_{\sim\lambda} = [v]_{\sim\lambda}$. So $(u,v) \in \sim_\lambda$. Then by the definition of $\sim_\lambda$, $\lambda(u) = \lambda(v)$. Thus, Syn(λ) recognizes λ, by Theorem 3.6.

Theorem 4.1. *Let λ be a l-fuzzy language over A. Then a monoid M recognizes λ if and only if Syn(λ) divides M.*

Proof. Assume that the monoid M recognizes λ. Then there exist a homomorphism $\varphi : A^* \to M$ and a l-fuzzy subset π of M such that $\lambda = \pi\varphi^{-1}$ where $\lambda(u) = \pi(\varphi(u))$, $u \in A^*$. Define a map ψ from $\varphi(A^*)$ to Syn(λ) by $\psi(\varphi(u)) = [u]_{\sim\lambda}$. Let $u, v \in A^*$ and let $\varphi(u) = \varphi(v)$. Then $(u,v) \in \ker\varphi$. By Theorem 3.6, we have $\lambda(u) = \lambda(v)$. So $[u]_{\sim\lambda} = [v]_{\sim\lambda}$. That is, $\psi(\varphi(u)) = \psi(\varphi(v))$, $u,v \in A^*$. Thus ψ is well defined. Also, we have

$$\begin{aligned}\psi(\varphi(u)\varphi(v)) &= \psi(\varphi(uv)) = [uv]_{\sim\lambda} \\ &= [u]_{\sim\lambda}[v]_{\sim\lambda} \\ &= \psi(\varphi(u))\psi(\varphi(v)),\end{aligned}$$

for all $u, v \in A^*$.

Clearly $\varphi(\varepsilon)$ is the identity in $\varphi(A^*)$, where ε is the empty word. Then $\psi[\varphi(\varepsilon)] = [\varepsilon]_{\sim\lambda}$ which is the identity in Syn(λ). Thus ψ is a homomorphism from $\varphi(A^*)$ into Syn(λ). Since $\varphi(A^*)$ is a submonoid of M, we see that Syn(λ) divides M.

Conversely assume that Syn(λ) divides a monoid M. We show that M recognizes λ. Since Syn(λ) divides M, there exist a submonoid M_1 of M and an onto homomorphism ψ from M_1 to Syn(λ). Define a map $\mu : A^* \to M_1$ by $\mu(u) = m$ if $\eta_\lambda(u) = \psi(m)$, for all $u \in A^*$ and $m \in M_1$. Let $u_1, u_2 \in A^*$ and let $\mu(u_1) = m_1$ and $\mu(u_2) = m_2$. Let $u_1 = u_2$, then $\eta_\lambda(u_1) = \eta_\lambda(u_2)$. Thus $\mu(u_1) = \mu(u_2)$. Hence μ is well defined. We have,

$$\begin{aligned}\eta_\lambda(u_1u_2) &= \eta_\lambda(u_1)\eta_\lambda(u_2) \\ &= \psi(m_1)\psi(m_2) \\ &= \psi(m_1m_2).\end{aligned}$$

Thus $\mu(u_1u_2) = m_1m_2 = \mu(u_1)\mu(u_2)$. Hence μ is a homomorphism from $A^* \to M_1$ and $\eta_\lambda = \psi\mu$. Since M_1 is a submonoid of M, there exists a homomorphism $\varphi : A^* \to M$.

Since Syn(λ) recognizes λ, there exists a l-fuzzy subset π_1 of Syn(λ) such that $\lambda = \pi_1\eta_\lambda^{-1}$ where $\lambda(u) = \pi_1(\eta_\lambda(u))$. Define a map π from M to l by $\pi(m) = (\pi_1\psi^{-1})(u)$ where $(\pi_1\psi^{-1})(u) = \pi_1(\psi(u))$, *if* $m \in M_1$. If $m \in M \setminus M_1$, then π is defined arbitrarily. Since ψ and π_1 are well defined, π is well defined. For $u \in A^*$, we have

$$\begin{aligned}\pi\varphi^{-1}(u) &= \pi(\varphi(u)) = (\pi_1\psi^{-1})(\varphi(u)) \\ &= \pi_1(\psi(\varphi(u))) = \pi_1(\psi(\mu(u))) \\ &= \pi_1(\eta_\lambda(u)) \\ &= (\pi_1\eta_\lambda^{-1})(u) = \lambda(u)\end{aligned}$$

Thus $\lambda = \pi\varphi^{-1}$. Hence λ is recognizable.

Corollary 4.2. *Syntactic monoid Syn(λ) is the minimal monoid recognizing the fuzzy language λ.*

It is well known that every monoid is the syntactic monoid of a fuzzy language (cf, [8]). Now we prove the case for l-fuzzy languages.

Theorem 4.3. *For every monoid M with $|M| \le |l|$, there exist a l-fuzzy language λ such that M is the syntactic monoid of λ.*

Proof. Let M be a monoid. Then there exist an alphabet A and an epimorphism $\varphi : A^* \to M$. Since $\ker\varphi$ is a congruence on A^*, $\ker\varphi$ partitions A^* into different equivalence classes (languages). Let $\{L_i\}_{i\in I}$ be languages in the partition determined by kerφ. Let $\{l_i\}_{i\in I}$ be pairwise distinct elements of the lattice l. Define a l-fuzzy language $\lambda : A^* \to l$ by

$$\lambda(u) = l_i \text{, if } u \in L_i$$

A map ϕ :Syn(λ) $\to M$ is defined by $\phi(\eta_\lambda(u)) = \varphi(u)$, $u \in A^*$. Let $u,v \in A^*$ and $\eta_\lambda(u) = \eta_\lambda(v)$. Then $(u,v) \in \sim_\lambda$. So $\lambda(u) = \lambda(v)$. Thus u and v belongs to some L_i. That is, $(u,v) \in$ kerφ. Hence we get, $\varphi(u) = \varphi(v)$. Therefore ϕ is well defined. We have

$$\begin{aligned}\phi(\eta_\lambda(u)\eta_\lambda(v)) &= \phi(\eta_\lambda(uv)) \\ &= \varphi(uv) \\ &= \varphi(u)\varphi(v) \\ &= \phi(\eta_\lambda(u))\phi(\eta_\lambda(v))\end{aligned}$$

Also $\phi(\eta_\lambda(u)) = \phi(\eta_\lambda(v))$ if and only if $\varphi(u) = \varphi(v)$. So $(u,v) \in$ kerφ. Hence $(u,v) \in L_i$ for some $i \in I$. Thus $\lambda(u) = \lambda(v)$. So $(u,v) \in \sim_\lambda$. That is, $[u]_{\sim\lambda} = [v]_{\sim\lambda}$. Hence ϕ is one to one.

Let $m \in M$. Since φ is onto, there exists some $u \in A^*$ such that $\varphi(u) = m$. So $\phi(\eta_\lambda(u)) = \varphi(u) = m$. Thus ϕ is an isomorphism from Syn(λ) onto M.

Example 4.4. *Let $l = (\{\emptyset, \{c\}, \{d\}, \{c,d\}\}, \cap, \cup)$ be the complete complemented distributive lattice and the monoid be $M = (Z_3, +_3)$. Here $|M| < |l|$. Let $A = \{a,b\}$. Then there exists a l-fuzzy language $\lambda : A^* \to l$ defined by*

$$\lambda(u) = \begin{cases} \emptyset & \text{if } |u| \equiv 0 \bmod 3 \\ \{c\} & \text{if } |u| \equiv 1 \bmod 3 \\ \{d\} & \text{if } |u| \equiv 2 \bmod 3. \end{cases}$$

Here $Syn(\lambda)$ is isomorphic to Z_3.

The following theorem presents the Myhill-Nerode theorem for l-fuzzy languages.

Theorem 4.5. *Let λ be a l-fuzzy language over an alphabet A. Then the following statements are equivalent*

(1) λ is recognizable.

(2) $\sim_\lambda$ has finite index.

Proof. **(1)** Assume that λ is recognizable. So λ is recognized by a finite monoid M. Then by Theorem 4.1, Syn(λ) divides M. That is, Syn(λ) is a homomorphic image of a submonoid of M. Thus Syn(λ) is finite. Hence $\sim_\lambda$ has finite index.

(2) Assume that $\sim_\lambda$ has finite index. So Syn(λ) is finite. Define a map π':Syn(λ) $\to l$ by $\pi'(\eta_\lambda(u)) = \lambda(u), u \in A^*$. Let $u_1, u_2 \in A^*$, and $\eta_\lambda(u_1) = \eta_\lambda(u_2)$. Then $[u_1]_{\sim\lambda} = [u_2]_{\sim\lambda}$. Thus $u_1 \sim_\lambda u_2$. Hence $\lambda(u_1) = \lambda(u_2)$. Thus π' is well defined and $\lambda(u) = \pi'(\eta_\lambda(u)) = (\pi'\eta_\lambda^{-1})(u)$ for all $u \in A^*$. Thus $\lambda = \pi'\eta_\lambda^{-1}$ and Syn(λ) recognizes λ by the syntactic homomorphism. Therefore λ is recognizable.

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