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Study Of Some Real Definite Integrals Over A Unit Triangle Via Adaptive Scheme

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Abstract- In this note, an effective quadrature rule with an adaptive scheme has been implemented over the triangular domain. The double transformations have been used to transform the triangular surface into a standard square space. The new quadrature rule has been employed in adaptive scheme taking 5 numerical texts for getting the improved result over the constituent rule and an error analysis has been proposed.

Index Terms- Mixed quadrature rule, Degree of precision, Maclaurin's series, Error bound, Adaptive quadrature scheme.

1. INTRODUCTION

The applications of mixed quadrature rule for the approximation of real integrals $I(h)=\int_u^v h(l)dl$ have been used by several authors. The symmetric Gaussian quadrature formula for integrating arbitrary functions of two variables over triangular region was proposed by [1], [2, 3], [4]. Next to them the same symmetric quadrature rules in polar co-ordinates was given by [5]. The symmetric integration formula with higher order precision up to degree ten was given by [6]. The researchers [7], [8] considered the product formula which was derived from Gaussian quadrature in single variable. The researcher [9] has proposed an alternative integration formula for triangular domain on the basis of finite element method. Lastly, the mixed quadrature on real definite integrals with finite element methods has been suggested by [10].

In this paper we adopt an application of mixed quadrature with adaptive scheme over the specific triangular domain $l,m\in[0,1]$ and $\{l+m\le 1\}$.Here we have implemented a transformation scheme to transform from (l,m) space to $\{(p,q):-1\le p,q\le 1\}$through a standard square $\alpha,\beta\in[0,1]$ space. A new rule of precision-7 has been obtained by mixing two constituent rules each of precision 5 .Our said rule is superior to that of Clenshaw-Curtis 5-point rule and a good agreement has been reached to that of exact result in adaptive scheme.

For a real integrable function h, an interval $[u,v]$ and a prescribed tolerance ε , it is desired to compute an approximation D to the integral $I=\int_u^v h(l)dl$ so that $|D-I|\le\varepsilon$.The basic principle for adaptive quadrature is the additive property of a definite integral of the form $D=E+F$ with the adaptive integration schemes [11,12,13] where $D=\int_u^v h(l)dl, E=\int_u^w h(l)dl, F=\int_w^v h(l)dl$ and w is any point between u and v In adaptive integration, the points at which the integrand is evaluated are so chosen in such a way that depends on the nature of the integrand. The basic concept for an approximation of an integral D is the sum of two integrals E and F within a specified tolerance ε .

Adaptive algorithm

The inputs to the algorithm are $0,1,n,h,r,\varepsilon$ .Where n is the number of intervals chosen and ε is the stopping criterion.

Step-1: An approximation

$$I=\iint_T h(l,m)dldm=\int_0^1 dm\int_0^{1-m} h(l,m)dl=\iint_R h(\alpha,\beta)d\alpha d\beta$$

$\alpha,\beta\in[0,1]$ is to be calculated.

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$\alpha,\beta\in[0,1]$ is to be calculated.

Step-2: Taking $c=\frac{0+1}{2}$, the original square is divided into 4 sub-squares each of side $c=\frac{1}{2}$ unit.

Step-3: For each individual square if each I_i satisfy $|I_i(h)-h(l,m)|\le\varepsilon$, then $I=I_1+I_2+I_3+I_4$

If not then go to step-2 i.e. $c=\frac{1}{2^2}$ unit and if $I_i, i=1,.....,4$ satisfy $|I_i(h)-h(l,m)|\le\varepsilon$, the process will be stopped and it is added with previous I_i .

Step-4: Continuing this process r -times for each sub-square of side $c=\frac{1}{2^r}$ and if $|I_i(h)-h(l,m)|\le\varepsilon$ then stop. Otherwise repeat step-3 until to get $|I-h(l,m)|\le\varepsilon$.

This paper is designed as follows. Sec-1 contains introduction, sec-2 bears the construction of quadrature rule over the domain T in adaptive environment, sec-3 contains the construction of mixed rule taking the two constituent rules $R_{L4}(h)$ and $R_{CC5}(h)$. The error analysis and error bound are worked out in sec-4.The numerical verification of our proposed rule is experimented on some suitable real integrals and the comparison with the constituent $R_{CC5}(h)$ rule in adaptive scheme sec-5.The conclusion follows in sec-6 .

2. GENERAL CONSTRUCTION OF QUADRATURE OVER A TRIANGULAR REGION

$$I(h)=\iint_T h(l,m)\,dl\,dm=\int_0^1 dm\int_0^{1-m} h(l,m)\,dl \qquad (2.1)$$

Where $l=\alpha,\quad m=(1-\alpha)\beta$

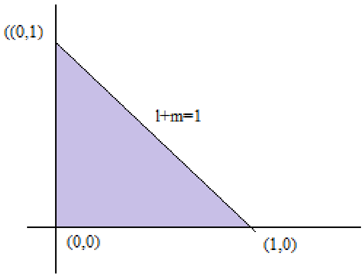


Fig-1 (Triangular region)

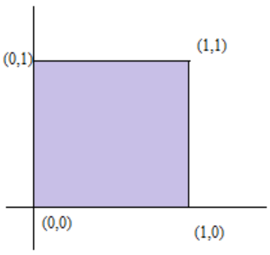


Fig-2 (Square region)

The Jacobian

$$J=\frac{\partial(l,m)}{\partial(\alpha,\beta)}=1-\alpha$$

$$Area=dl\,dm=J\,d\alpha\, d\beta$$

Now eqn (2.1) becomes,

$$I(h)=\int_0^1\int_0^1 h(\alpha,(1-\alpha)\beta)(1-\alpha)\,d\alpha\, d\beta \qquad (2.2)$$

Taking the substitution

$$\alpha=\frac{1+p}{2},\ \beta=\frac{1+q}{2} \qquad (2.3)$$

$$d\alpha d\beta=\frac{\partial(\alpha,\beta)}{\partial(p,q)}dpdq=\frac{1}{4}dpdq \qquad (2.4)$$

Now using eqn (2.3) and eqn (2.4) in eqn (2.2),

$$I(h)=\int_{-1}^1\int_{-1}^1 h\left(\frac{1+p}{2},\frac{(1-p)(1+q)}{4}\right)\left(\frac{1-p}{8}\right)dp\,dq \qquad (2.5)$$

Where $p,q\in[-1,1]$

3. MIXED RULE

In this section a new (mixed) rule has been established for approximation of

$$I(h)=\int_{-1}^1\int_{-1}^1 h(l,m)\,dl\,dm \qquad (3.1)$$

Clenshaw-Curtis five point rule

$I(h) \cong R_{CC5}(h)$

$$= \frac{1}{225}\left[\begin{array}{l} \left\{\begin{array}{l} h(1,1)+h(1,-1)+8h\left(1,\frac{1}{\sqrt{2}}\right) \\ +8h\left(1,-\frac{1}{\sqrt{2}}\right)+12h(1,0) \end{array}\right\}+ \\ \left\{\begin{array}{l} h(-1,1)+h(-1,-1)+8h\left(-1,\frac{1}{\sqrt{2}}\right) \\ +8h\left(-1,-\frac{1}{\sqrt{2}}\right)+12h(-1,0) \end{array}\right\}+ \\ 8\left\{\begin{array}{l} h\left(\frac{1}{\sqrt{2}},1\right)+h\left(\frac{1}{\sqrt{2}},-1\right) \\ +8h\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)+8h\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) \\ +12h\left(\frac{1}{\sqrt{2}},0\right) \end{array}\right\} \\ +8\left\{\begin{array}{l} h\left(-\frac{1}{\sqrt{2}},1\right)+h\left(-\frac{1}{\sqrt{2}},-1\right) \\ +8h\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) \\ +8h\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)+12h\left(-\frac{1}{\sqrt{2}},0\right) \end{array}\right\} \\ +12\left\{\begin{array}{l} h(0,1)+h(0,-1)+8h\left(0,\frac{1}{\sqrt{2}}\right) \\ +8h\left(0,-\frac{1}{\sqrt{2}}\right)+12h(0,0) \end{array}\right\} \end{array}\right] \quad (3.2)$$

Lobatto four point rule

$I(h) \cong R_{L4}(h)$

$$= \frac{1}{36}\left[\begin{array}{l} \left\{\begin{array}{l} h(1,\ 1)+5h\left(1,\ \frac{1}{\sqrt{5}}\right)+ \\ 5h\left(1,-\frac{1}{\sqrt{5}}\right)+h(1,\ -1) \end{array}\right\}+ \\ 5\left\{\begin{array}{l} h\left(\frac{1}{\sqrt{5}},1\right)+5h\left(\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right) \\ +5h\left(\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right)+h\left(\frac{1}{\sqrt{5}},-1\right) \end{array}\right\}+ \\ 5\left\{\begin{array}{l} h\left(-\frac{1}{\sqrt{5}},1\right)+5h\left(-\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right) \\ +5h\left(-\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right)+h\left(-\frac{1}{\sqrt{5}},-1\right) \end{array}\right\}+ \\ \left\{\begin{array}{l} h(-1,\ 1)+\ 5h\left(-1,\ \frac{1}{\sqrt{5}}\right) \\ +\ 5h\left(-1,\ -\frac{1}{\sqrt{5}}\right)+h(-1,\ -1) \end{array}\right\} \end{array}\right] \quad (3.3)$$

Expanding each term of eqn (3.2) and eqn (3.3) using Maclaurin's series,

$$I(h) = R_{CC5}(h) + E_{CC5}(h) \quad (3.4)$$

$$I(h) = R_{L4}(h) + E_{L4}(h) \quad (3.5)$$

Where

$$\begin{aligned} R_{CC5}(h) &= 4h_{0,0}(0,0) + \frac{2}{3}\left[h_{2,0}(0,0) + h_{0,2}(0,0)\right] \\ &+ \frac{1}{30}\left[h_{4,0}(0,0) + h_{0,4}(0,0)\right] \\ &+ \frac{1}{9}h_{2,2}(0,0) + \frac{1}{180}\left[h_{4,2}(0,0) + h_{2,4}(0,0)\right] \\ &+ \frac{8}{15\times 6!}\left[h_{6,0}(0,0) + h_{0,6}(0,0)\right] + \\ &+ \frac{1}{3600}h_{4,4}(0,0) + \frac{224}{45\times 8!}\begin{bmatrix} h_{6,2}(0,0) \\ +h_{2,6}(0,0) \end{bmatrix} \\ &+ \frac{2}{5\times 8!}\left[h_{8,0}(0,0) + h_{0,8}(0,0)\right] + \ldots \end{aligned} \quad (3.6)$$

$$R_{L4}(h)=4h_{0,0}(0,0)+\frac{2}{3}\begin{bmatrix}h_{2,0}(0,0)\\+h_{0,2}(0,0)\end{bmatrix}$$
$$+\frac{1}{30}\left[h_{4,0}(0,0)+h_{0,4}(0,0)\right]$$
$$+\frac{1}{9}h_{2,2}(0,0)+\frac{1}{180}\left[h_{4,2}(0,0)+h_{2,4}(0,0)\right] \quad (3.7)$$
$$+\frac{52}{75\times 6!}\left[h_{6,0}(0,0)+h_{0,6}(0,0)\right]+$$
$$+\frac{1}{3600}h_{4,4}(0,0)+\frac{1456}{225\times 8!}\begin{bmatrix}h_{6,2}(0,0)\\+h_{2,6}(0,0)\end{bmatrix}$$
$$+\frac{84}{125\times 8!}\left[h_{8,0}(0,0)+h_{0,8}(0,0)\right]+\ldots$$

We can write eqn (3.1) using Maclaurin's expansion

$$I(h)=4h_{0,0}(0,0)+\frac{2}{3}\begin{bmatrix}h_{2,0}(0,0)\\+h_{0,2}(0,0)\end{bmatrix}$$
$$+\frac{1}{30}\left[h_{4,0}(0,0)+h_{0,4}(0,0)\right]$$
$$+\frac{1}{9}h_{2,2}(0,0)+\frac{1}{180}\begin{bmatrix}h_{4,2}(0,0)\\+h_{2,4}(0,0)\end{bmatrix} \quad (3.8)$$
$$+\frac{4}{7!}\left[h_{6,0}(0,0)+h_{0,6}(0,0)\right]+$$
$$+\frac{1}{3600}h_{4,4}(0,0)+\frac{1}{7560}\begin{bmatrix}h_{6,2}(0,0)\\+h_{2,6}(0,0)\end{bmatrix}$$
$$+\frac{4}{9!}\left[h_{8,0}(0,0)+h_{0,8}(0,0)\right]+\ldots$$

$$E_{CC5}(h)=I(h)-R_{CC5}(h)$$
$$=\frac{4}{105\times 6!}\left[h_{6,0}(0,0)+h_{0,6}(0,0)\right]$$
$$+\frac{16}{45\times 8!}\left[h_{6,2}(0,0)+h_{2,6}(0,0)\right] \quad (3.9)$$
$$+\frac{2}{45\times 8!}\left[h_{8,0}(0,0)+h_{0,8}(0,0)\right]$$

$$E_{L4}(h)=I(h)-R_{L4}(h)$$
$$=-\frac{64}{75\times 7!}\left[h_{6,0}(0,0)+h_{0,6}(0,0)\right]$$
$$-\frac{256}{25\times 9!}\begin{bmatrix}h_{6,2}(0,0)\\+h_{2,6}(0,0)\end{bmatrix} \quad (3.10)$$
$$-\frac{256}{125\times 9!}\left[h_{8,0}(0,0)+h_{0,8}(0,0)\right]$$

Now multiplying $\left(\frac{16}{5}\right)$ in eqn (3.4) and adding eqn (3.4) and eqn (3.5) we get,

$$I(h)=R_{CC5L4}(h)+E_{CC5L4}(h) \quad (3.11)$$

Where

$$R_{CC5L4}(h)=\frac{1}{21}\left[16R_{CC5}(h)+5R_{L4}(h)\right] \quad (3.12)$$
$$E_{CC5L4}(h)=\frac{1}{21}\left[16E_{CC5}(h)+5E_{L4}(h)\right] \quad (3.13)$$

Now, $E_{CC5L4}(h)$ can be obtained by substituting eqn (3.9) and eqn (3.10) in eqn (3.13),

$$E_{CC5L4}(h)=-\frac{32}{1575\times 8!}\begin{bmatrix}h_{8,0}(0,0)\\+h_{0,8}(0,0)\end{bmatrix} \quad (3.14)$$

4. ERROR ESTIMATION

Theorem-4.1 The error $E_{CC5L4}(h)=I-R_{CC5L4}(h)$ and $\left|E_{CC5L4}(h)\right|\le\frac{128K}{2205\times 6!}$ where

$$K=\max_{\substack{-1\le x\le 1\\-1\le y\le 1}}\left|h_{7,0}(l,*)+h_{0,7}(*,m)\right|$$

Proof: We have

$$E_{CC5}(h)=\frac{4}{105\times 6!}\left[h_{6,0}(\eta_2,0)+h_{0,6}(0,\eta_2)\right]$$

$$E_{L4}(h)=-\frac{64}{525\times 6!}\left[h_{6,0}(\eta_1,0)+h_{0,6}(0,\eta_1)\right]$$

where $\eta_1,\eta_2\in[-1,1]$

$$E_{CC5L4}(h)=\frac{1}{21}\left[16E_{CC5}(h)+5E_{L4}(h)\right]$$
$$=\frac{64}{2205\times 6!}\begin{bmatrix}h_{6,0}(\eta_2,0)+h_{0,6}(0,\eta_2)-h_{6,0}(\eta_1,0)\\-h_{0,6}(0,\eta_1)\end{bmatrix}$$
$$=\frac{64}{2205\times 6!}\left[\int_{\eta_1}^{\eta_2}h_{7,0}(l,0)\,dl+\int_{\eta_1}^{\eta_2}h_{0,7}(0,m)\,dm\right]$$
$$=\frac{64}{2205\times 6!}\int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2}\left[h_{7,0}(l,*)+h_{0,7}(*,m)\right]dl\,dm$$

$$\left|E_{CC5L4}(h)\right|\le\frac{64K}{2205\times 6!}\left|\eta_2-\eta_1\right|$$

for $\left|\eta_2-\eta_1\right|\le 2$ (C.Conte and D.Boor [11])

$$\left|E_{CC5L4}(h)\right| \le \frac{128K}{2205\times 6\,!}$$

Where $K = \max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}} \left|h_{7,0}(l,*) + h_{0,7}(*,m)\right|$

K gives the truncational error bound as η_1, η_2 are unknown points in $[-1,1]$

5. Numerical verification

Here we have taken 5 tests for our purpose. The integrals are

$$I_1 = \int_0^1\int_0^{1-l} \sin(l+m)\,dm\,dl$$

$$I_2 = \int_0^1\int_0^{1-l} e^{l+m}\,dm\,dl$$

$$I_3 = \int_0^1\int_0^{1-l} \cosh(l+m)\,dm\,dl$$

$$I_4 = \int_0^1\int_0^{1-l} \cos^2(l+m)\,dm\,dl\,,$$

$$I_5 = \int_0^1\int_0^{1-l} e^{l}\cos m\,dm\,dl\,.$$

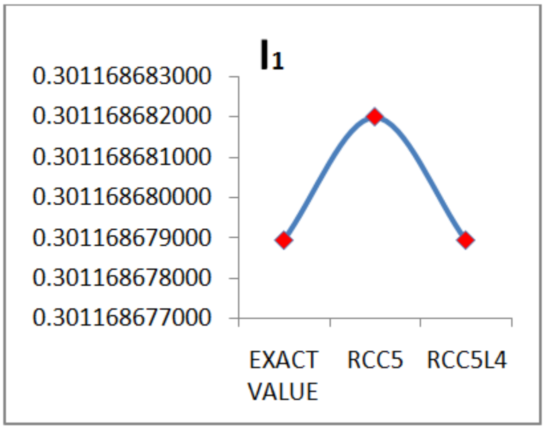
$Table-*$

(Numerical test for $R_{CC5}(h)$ and $R_{CC5L4}(h)$ with stopping criterion ε)

Exact value	$R_{CC5}(h)$ *by adaptive method*
I_1=0.301168678939757	0.301168681730568
I_2=1.000000000000000	1.000000000027503
I_3=0.632120558828558	0.632120558803135
I_4=0.300306002138028	0.300305994129496
I_5=0.668254268891505	0.668254274236139

$Table-*$

No of Intervals for $R_{CC5}(h)$	$R_{CC5L4}(h)$ *by adaptive method*	*No of Intervals for* $R_{CC5L4}(h)$	*absolute error* (ε)
2	0.3011686 78941508	1	$\varepsilon_1 = 0.000000002$
2	0.9999999 99997136	1	$\varepsilon_2 = 0.00000000002$
2	0.6321205 58828558	1	$\varepsilon_3 = 0.00000000002$
2	0.3003060 02056161	1	$\varepsilon_4 = 0.000000008$
3	0.6682542 68913063	1	$\varepsilon_5 = 0.000000005$



Comparison of Exact value with $R_{CC5}(h)$ and $R_{CC5L4}(h)$ of I_1.

6. CONCLUSION

The new (mixed) rule $R_{CC5L4}(h)$ has made an agreement with that of exact value for different integrals as well as superior to $R_{CC5}(h)$ rule from $Table-*$ and the graphs for I_1. The error and the error bound have been clearly reflected for our motto.

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