(1, 2) * - $\alpha^*$ - Compact Spaces in Bitopological Spaces

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Abstract- In this paper, (1, 2)* - $\alpha^*$ - isolated point, (1, 2)* - $\alpha^*$- compact spaces , (1, 2)* - $\alpha^*$ - countably compact spaces and sequentially (1, 2) * - $\alpha^*$ - compact spaces in bitopological spaces are introduced and its properties are investigated.

Keywords - (1, 2)* - $\alpha^*$ - isolated point , (1, 2) * - $\alpha^*$ - compact spaces , (1, 2)* - $\alpha^*$ - countably compact spaces and sequentially (1, 2) * - $\alpha^*$ - compact spaces.

1. INTRODUCTION

The study about a class of compact space called as S-closed space using semi open cover was first initiated by DI Maio and Noiri [4]. The notion of locally S-closed space was investigated by Noiri[7]. The class of compact space namely GO-compact space and GO-connected space using g-open cover was introduced by Balachandran, Sundaram and Maki[1]. Pauline Mary Helen, Ponnuithai Selvarani, Veronica Vijayan, Puthitha Tharani[8] studied about $g^*$ compact space and $g^*$ compact space modulo I space Mohana and Arockiariani[6] introduced a class of (1, 2)*- $\pi$ $ga^*$-compact spaces and (1, 2)*- $\pi$ $ga^*$-connected spaces using (1, 2)*- $\pi$ $ga^*$-open sets in bitopological spaces.

In this paper, we introduce a new class of (1, 2)*- $\alpha^*$ - compact space using (1, 2)*- $\alpha^*$ open sets and investigate their properties. Further, we study (1, 2)*- $T_{\alpha^*}$- space , (1, 2)*- $\overline{T_{\alpha^*}}$- space and their properties.

2. PRELIMINARIES

Throughout this paper, X and Y denote the bitopological spaces $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ respectively, on which no separation axioms are assumed.

Definition 2.1. [5] A subset S of a bitopological space X is said to be $\tau_{1,2}$-open if $S\subseteq A\cup B$ where $\tau_1 \in A$ and $\tau_2 \in B$. A subset S of X is said to be (i) $\tau_{1,2}$-closed if the complement of S is $\tau_{1,2}$-open. (ii) $\tau_{1,2}$-closed if S is both $\tau_{1,2}$-open and $\tau_{1,2}$-closed.

Definition 2.2. [5] Let S be a subset of the bitopological space X. Then the $\tau_{1,2}$- interior of S denoted by $\tau_{1,2}$-int(S) is defined by $\cup\{G; G\subseteq S$ and G is $\tau_{1,2}$- open$\}$ and $\tau_{1,2}$- closure of S denoted by $\tau_{1,2}$-cl(S) is defined by $\cap\{F; F\subseteq F$ and F is $\tau_{1,2}$-closed$\}$.

Definition 2.3. A subset A of a bitopological space X is said to be

i) (1, 2)*-regular open [5] if $A=\tau_{1,2}$-int $(\tau_{1,2}$-cl(A)).

ii) (1, 2)*-a-open[5] if $A\subseteq\tau_{1,2}$-int $(\tau_{1,2}$-cl($\tau_{1,2}$-int(A))).

iii) $(1, 2)^{\pi}$ - $\alpha^*$ - closed[2] if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A\subseteq U$ and U is $(1, 2)^{\pi}$ - $\alpha^*$ - open set in X.

Definition 2.4. A map $f:X \rightarrow Y$ is called

(i) (1, 2)*- $\alpha^*$-continuous [3] if $f^{1}(V)$ is (1, 2)*- $\alpha^*$-closed in X for every $\sigma_{1,2}$-closed set V of Y.

(ii) (1, 2)*- $\alpha^*$- irresolute[3] if $f^{1}(V)$ is (1, 2)*- $\alpha^*$-closed in X for every (1, 2)*- $\alpha^*$-closed V of Y.

Definition 2.5. [8] A map $f:X \rightarrow Y$ is called $g^*$- irresolute if $f(V)$ is $g^*$-open in Y whenever U is $g^*$-open in X.

3. (1, 2) * - $\alpha^*$ - COMPACT SPACES

Definition 3.1. A collection $\{A_i; i \in I\}$ of (1, 2)*- $\alpha^*$-open sets in a bitopological space X is called a (1, 2)*- $\alpha^*$-open cover of a subset B if $B \subseteq \bigcup\{A_i; i \in I\}$.

Definition 3.2. A bitopological space X is called (1, 2)*- $\alpha^*$-compact, if every (1, 2)*- $\alpha^*$-open covering of X contains a finite subcollection that also covers X. A subset A of X is said to be (1, 2)*- $\alpha^*$-compact if every (1, 2)*- $\alpha^*$-open covering of A contains a finite subcollection that also covers A.

Definition 3.3. A subset B of a bitopological space X is said to be (1, 2)*- $\alpha^*$-compact relative to X, if for
every collection \{A_i: i \in I\} of (1, 2)^*-α*-open subsets of X such that B \subseteq \bigcup \{A_i: i \in I\} then, there exists a finite subset I_0 of I such that B \subseteq \bigcup \{A_i: i \in I_0\}.

**Definition 3.4.** A subset B of a bitopological space X is said to be (1, 2)^*-α*-compact if B is (1, 2)^*-α*-compact as the subset of X.

**Remark 3.5.** Any topological space having only finitely many points is necessarily (1, 2)^*-α*-compact and (1, 2)^*α*-compact.

**Theorem 3.6.** A (1, 2)^*α*-closed subset of (1, 2)^*α*-compact space is (1, 2)^*α*-compact.

**Proof.** Let A be a (1, 2)^*α*-closed subset of (1, 2)^*α*-compact space \((X, \tau_1, \tau_2)\) and \(\{U_a\}_{a \in \Delta}\) be a (1,2)^*α*-open cover for A. Then, \(\{\{U_a\}_{a \in \Delta}, (X - A)\}\) is a (1, 2)^*α*-open cover for X. Since X is (1, 2)^*α*-compact, there exists \(\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta\) such that, therefore, which proves A is (1, 2)^*α*-compact.

**Theorem 3.7.** A (1, 2)^*α*-closed subset of (1, 2)^*α*-compact space is (1, 2)^*α*-compact relative to X.

**Proof.** Let A be a (1, 2)^*α*-closed subset of a (1, 2)^*α*-compact space X. Then \(X'\) is (1, 2)^*α*-open in X. Let S be a cover of A by (1, 2)^*α*-open sets in X. Then, \(\{S, X'\}\) is a (1, 2)^*α*-open cover of X. Since X is (1, 2)^*α*-compact, it has a finite subcover, say \(\{G_1, G_2, \ldots, G_n\}\). If this subcover contains \(X'\), we discard it. Otherwise, leave the subcover as it is. Thus we have obtained a finite (1, 2)^*α*-open subcover of A and so A is (1, 2)^*α*-compact relative to X.

**Theorem 3.8.**

(i) If \(f: X \to Y\) is (1, 2)^*α*-continuous image of a (1, 2)^*α*-compact space, then (1, 2)^*α*-compact.

(ii) If a map \(f: X \to Y\) is (1, 2)^*α*- irresolute and a subset B is (1, 2)^*α*-compact relative to X, then the image \(f(B)\) is (1, 2)^*α*-compact relative to Y.

**Proof.**

(i) Let \(f: X \to Y\) be a (1, 2)^*α*-continuous map from a (1, 2)^*α*-compact space X onto a bitopological space Y. Let \(\{A_i: i \in I\}\) be an open cover of Y. Then \(f^{-1}(A_i): i \in I\) is a open cover of X. Since X is (1, 2)^*α*-compact, it has a finite subcover, say \(\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}\). Since f is onto, \(\{A_1, A_2, \ldots, A_n\}\) is a σ\(_{1,2}\)-open cover of Y and so Y is (1, 2)^*α*-compact.

(ii) Let \(\{A_i: i \in I\}\) be any collection of (1, 2)^*α*-open subsets of Y such that \(f(B) \subseteq \bigcup \{A_i: i \in I\}\). Then, B \subseteq \bigcup \{f(A_i): i \in I\}. By using assumptions, there exists a finite subset I_0 of I such that B \subseteq \bigcup \{f(A_i): i \in I_0\}. Therefore, we have \(f(B) \subseteq \bigcup \{A_i: i \in I_0\}\) which shows that \(f(B)\) is (1, 2)^*α*-compact relative to Y.

**Definition 3.9.** A map \(f: X \to Y\) is called (1, 2)^*α*-resolute if \(f(V)\) is (1, 2)^*α*-open in Y whenever U is (1, 2)^*α*-open in X.

**Definition 3.10.** A map \(f: X \to Y\) is called strongly-(1, 2)^*α*-continuous if \(f^{-1}(V)\) is \(\tau_{12}\)-closed for every collection \(\{f^{-1}(A_i): i \in I\}\) in X for every (1, 2)^*α*-closed \((\tau_{12}, \tau_{*}\alpha*)\) set V of Y.

**Definition 3.11.** A topological space \((X, \tau_1, \tau_2)\) is said to be (1, 2)^*α*-compact if every pair of distinct points \(x, y\) in X, there exists disjoint \((1, 2)^*\alpha*- open sets U and V in X such that \(x \in U, y \notin U\) and \(x \notin V, y \in V\).

**Definition 3.12.** A topological space \((X, \tau_1, \tau_2)\) is said to be (1, 2)^*α*- compact space for every pair of distinct points \(x, y\) in X, there exists disjoint \((1, 2)^*\alpha*- open sets U and V in X such that \(x \in U\) and \(y \in V\).

**Definition 3.13.** A collection \(\zeta\) of subsets of X is said to have finite intersection property if for every subcollection \(\{C_1, C_2, \ldots, C_n\}\) of \(\zeta\), the intersection \(\bigcap C_1 \cap C_2 \cap \ldots \cap C_n\) is non-empty.

**Theorem 3.14.** If a map \(f: X \to Y\) is strongly-(1, 2)^*α*-continuous from a (1, 2)^*α*-compact space X onto a bitopological space Y, then Y is (1, 2)^*α*-compact.

**Proof.** Let \(\{A_i: i \in I\}\) be a (1, 2)^*α*-open cover of \(Y\). Then \(\{f^{-1}(A_i): i \in I\}\) is a \(\tau_{12}\)-open cover of X. As f is strongly-(1, 2)^*α*-continuous. Since X is (1, 2)^*α*-compact, it has a finite subcover, say \(\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}\). Since f is onto, \(\{A_1, A_2, \ldots, A_n\}\) is a finite (1, 2)^*α*-open cover of Y and so Y is (1, 2)^*α*-compact.

**Theorem 3.15.** Let \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) be two topological spaces and \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be a function. Then

1. \(f\) is (1, 2)^*α*- irresolute and A is a (1, 2)^*α*-compact subset of X \(\Rightarrow f(A)\) is (1, 2)^*α*-compact subset of Y.

2. \(f\) is one to one \((1, 2)^*\alpha*-resolute and B is a (1, 2)^*α*-compact subset of Y \(\Rightarrow f(B)\) is a (1, 2)^*α*-compact subset of X.

**Proof.** (1) and (2) Obvious from the definitions.
Theorem: 3.16 A topological space \( (X, \tau_1, \tau_2) \) is (1, 2)*-\( \alpha \)-compact if and only if for every collection \( \zeta \) of (1, 2)*-\( \alpha \)-closed sets in \( X \) having finite intersection property, \( \bigcap_{c \in C} C \) of all elements of \( \zeta \) is non-empty.

Proof: Let \( (X, \tau_1, \tau_2) \) is (1, 2)*-\( \alpha \)-compact and \( \zeta \) be a collection of (1, 2)*-\( \alpha \)-closed sets with finite intersection property. Suppose \( \bigcap_{c \in C} C = \emptyset \), then \( \bigcap_{c \in C} (X - C) = X \). Therefore, \( \{X - C\}_{c \in C} \) is a (1, 2)*-\( \alpha \)-open cover for \( X \). Then there exists \( C_1, C_2, \ldots, C_n \in \zeta \) such that \( \bigcup_{i=1}^{n} (X - C_i) = X \).

Therefore, \( \bigcap_{i=1}^{n} C_i = \emptyset \) which is a contradiction.

Therefore, \( \bigcap_{c \in C} C \neq \emptyset \). Conversely, assume the hypothesis given in the statement. To prove \( X \) is (1, 2)*-\( \alpha \)-compact. Let \( \{U_a\}_{a \in A} \) be a (1, 2)*-\( \alpha \)-open cover for \( X \). Then \( \bigcup_{a \in A} U_a = X \Rightarrow \bigcap_{a \in A} (X - U_a) = \emptyset \). By the hypothesis, there exists \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that \( \bigcap_{i=1}^{n} X - U_{\alpha_i} = \emptyset \). Therefore, \( \bigcup_{i=1}^{n} U_{\alpha_i} = X \). Therefore, \( X \) is (1, 2)*-\( \alpha \)-compact.

Corollary 3.17: Let \( (X, \tau_1, \tau_2) \) be a (1, 2)*-\( \alpha \)-compact space and let \( C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq C_{n+1} \supseteq \ldots \) be a nested sequence of non-empty (1, 2)*-\( \alpha \)-closed sets in \( X \). Then \( \bigcap_{c \in C} C \) is non-empty.

Proof: Obviously \( \{C_n\}_{n \in \mathbb{N}} \) has finite intersection property. Therefore by Theorem 3.15, \( \bigcap_{c \in C} C \) is non-empty.

Theorem 3.18. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a function, then

1) \( f \) is (1, 2)*-\( \alpha \)-continuous, onto and \( X \) is (1, 2)*-\( \alpha \)-compact \( \Rightarrow \) \( Y \) is compact.
2) \( f \) is continuous, onto and \( X \) is (1, 2)*-\( \alpha \)-compact \( \Rightarrow \) \( Y \) is compact.
3) \( f \) is (1, 2)*-\( \alpha \)-irresolute, onto and \( X \) is (1, 2)*-\( \alpha \)-compact \( \Rightarrow \) \( Y \) is (1, 2)*-\( \alpha \)-compact.
4) \( f \) is strongly (1, 2)*-\( \alpha \)-irresolute, onto and \( X \) is (1, 2)*-\( \alpha \)-compact \( \Rightarrow \) \( Y \) is (1, 2)*-\( \alpha \)-compact.
5) \( f \) is (1, 2)*-\( \alpha \)-open, bijection and \( Y \) is (1, 2)*-\( \alpha \)-compact \( \Rightarrow \) \( X \) is compact.
6) \( f \) is open, bijection and \( Y \) is (1, 2)*-\( \alpha \)-compact \( \Rightarrow \) \( X \) is compact.
7) \( f \) is (1, 2)*-\( \alpha \)-resolute, bijection and \( Y \) is (1, 2)*-\( \alpha \)-compact \( \Rightarrow \) \( X \) is (1, 2)*-\( \alpha \)-compact.

Proof: (1): Let \( \{U_a\}_{a \in A} \) be an open cover for \( X \). Then \( \{f^{-1}(U_a)\}_{a \in A} \) is a (1, 2)*-\( \alpha \)-open cover for \( X \). Since \( X \) is (1, 2)*-\( \alpha \)-compact, there exists \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that \( X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i}) \).

Therefore, \( Y \) is compact. Proof for (2) to (7) are similar above.

4. (1, 2)*-\( \alpha \)-COUNTABLY COMPACT SPACE

Definition: 4.1. A subset \( A \) of a topological space \( (X, \tau_1, \tau_2) \) is said to be (1, 2)*-\( \alpha \)-countably compact space if every countable (1, 2)*-\( \alpha \)-open covering of \( A \) has a finite subcover.

Remark 4.2. Every (1, 2)*-\( \alpha \)-compact space is (1, 2)*-\( \alpha \)-countably compact space.

Definition: 4.3: Let \( (X, \tau_1, \tau_2) \) be a topological space and \( x \in X \). Every (1, 2)*-\( \alpha \)-open set containing \( x \) is said to be a (1, 2)*-\( \alpha \)-neighbourhood of \( x \).

Definition: 4.4: Let \( A \) be a subset of a topological space \( (X, \tau_1, \tau_2) \). A point \( x \in X \) said to be (1, 2)*-\( \alpha \)-limit point of \( A \) if every (1, 2)*-\( \alpha \)-neighbourhood of \( x \) contains a point of \( A \) other than \( x \).

Theorem 4.5: In a (1, 2)*-\( \alpha \)-countably compact topological space every infinite subset has a (1, 2)*-\( \alpha \)-limit point.

Proof: Let \( (X, \tau_1, \tau_2) \) be (1, 2)*-\( \alpha \)-countably compact space. Suppose that there exists an infinite subset which has no (1, 2)*-\( \alpha \)-limit point. Let \( B = \{a_n \mid n \in \mathbb{N} \} \) be a countable subset of \( A \). Since \( B \) has no (1, 2)*-\( \alpha \)-limit point of \( B \), there exists a \( (1, 2)*-\( \alpha \)-neighbourhood \( U_n \) of \( a_n \) such that \( B \cap U_n = \{a_n\} \). Now \( \{U_n\} \) is a (1, 2)*-\( \alpha \)-open cover for \( B \). Since \( B' = \{(1, 2)*-\( \alpha \)- open, \( \{U_n\}_{n \in \mathbb{N}} \} \) is a countable (1, 2)*-\( \alpha \)-open cover for \( X \). But it has no finite subcover which is a contradiction, since \( X \) is (1, 2)*-\( \alpha \)-countably compact. Therefore every infinite subset of \( X \) has a (1, 2)*-\( \alpha \)-limit point.
Corollary 4.6: In a compact topological space every infinite subset has a limit point.

Proof follows from theorem 4.5, since every \((1, 2)^{\ast}\)-compact is \((1, 2)^{\ast}\)-countably compact.

**Theorem 4.7:** A \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-closed subset of \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-countably compact space is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-countably compact. Proof is similar to theorem 3.6.

**Definition:** 4.8 In a topological space \((X, \tau, \tau_1, \tau_2)\), a point \(x \in X\) is said to be \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-isolated point of \(A\) if every \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-open set containing \(x\) contains no point of \(A\) other than \(x\).

**Theorem 4.9:** Let \(X\) be a non-empty \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-compact \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-\(T_{1}\) space. If \(A\) has no \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-isolated points, then \(X\) is uncountable.

**Proof:** Let \(x_n \in X\). Choose a point \(y\) of \(X\) different from \(x\). This is possible since \(\{x_n\}\) is not a \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-isolated point. Since \(X\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-\(T_{1}\) space, \(U_1\) and \(V_1\) such that \(U_1 \cap V_1 = \varnothing; x \in U_1, y \in V_1\). Therefore \(V_1\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-open and \(x_n \not\in g^{\ast} \text{cl}(V_1)\). By repeating the same process with \(V_1\) in the place of \(X\) and \(x_n\) in the place of \(y\) we get a point \(x \neq x_n\) and \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-open \(V_2\) set such that \(V_2\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-open and \(x_n \not\in (1, 2)^{\ast}\)-\(\alpha^{\ast}\)-cl\((V_2)\). In general, \(V_n\) which is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-open and non-empty, \(x_n \not\in (1, 2)^{\ast}\)-\(\alpha^{\ast}\)-cl\((V_n)\).

Hence we get a nested sequence of \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-closed sets such that \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-cl\((V_n)\) \(\supseteq (1, 2)^{\ast}\)-\(\alpha^{\ast}\)-cl\((V_{n+1})\) \(\supseteq \ldots\ldots\).

Since \(X\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-\(\alpha^{\ast}\)-compact, \(\cap(1, 2)^{\ast}\)-\(\alpha^{\ast}\)-cl\((V_n)\) \(\neq \varnothing\). Therefore, there exists \(x \in \cap(1, 2)^{\ast}\)-\(\alpha^{\ast}\)-cl\((V_n)\) \(\neq \varnothing\). But \(x \neq x_n\) for every \(n\), since \(x_n \not\in (1, 2)^{\ast}\)-\(\alpha^{\ast}\)-cl\((V_n)\).

**Theorem 4.11:** In a \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-\(T_1\) space, if every infinite subset has a \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-limit point, the \(X\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-countably compact.

**Proof:** Let every infinite subset \(S\) of \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-limit point. To prove \(X\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-countably compact. If not there exists a \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-open cover \(\{U_n\}\) such that it has no finite sub cover. Since \(U_1 \neq X\), there exists \(x_n \not\in U_1\); since \(X \neq U_1 \cup U_2\), there exists \(x_2 \not\in U_1 \cup U_2\). Proceeding like this there exists \(x_2 \not\in U_1 \cup U_2 \cup \ldots\ldots \cup U_n\). \(U \neq \{x_n\}\) is an infinite set. If \(x \in X\), then \(x \neq U_n\) for some \(n\). But \(x_n \not\in U_n\) for all \(k \geq n\). \(U_n - \{x_1, x_2, x_3, \ldots, x_{n-1}\}\) is a \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-open set (since \(X\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-\(T_1\)) containing \(x\) which does not have a point of \(A\) other than \(x\). Therefore, \(x\) is not a limit point of \(A\) which is a contradiction.

**Theorem 4.12** A topological space \((X, \tau, \tau_1, \tau_2)\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-\(\alpha^{\ast}\)-countably compact if and only if for every countable collection \(\zeta\) of \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-closed sets in \(X\) having finite intersection property, all elements of \(\zeta\) is non-empty.

**Proof:** Similar to the proof of theorem 3.15.

**Corollary 4.13:** \(X\) is \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-\(\alpha^{\ast}\)-countably compact if and only if every nested sequence of \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-closed non-empty sets \(C_1 \supseteq C_2 \supseteq \ldots\ldots\) has a non-empty intersection.

**Proof:** Obviously \(\{C_n\}_{n=0}^{\infty}\) has finite intersection property. Therefore, by theorem 4.10, \(\cap C_n\) is non-empty.

5. SEQUENTIALLY \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-COMPACT SPACE

**Definition:** 5.1 A subset \(A\) of a topological space \((X, \tau, \tau_1, \tau_2)\) is said to be sequentially \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-compact space if every sequence in \(A\) contains a subsequence which \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-converges to some point in \(A\).

**Theorem 5.2:** A finite subset \(A\) of a topological space \((X, \tau, \tau_1, \tau_2)\) is sequentially \((1, 2)^{\ast}\)-\(\alpha^{\ast}\)-compact.

**Proof:** Let \(\{x_n\}\) be an arbitrary sequence in \(X\). Since \(X\) is finite, at least one element of the sequence say \(x_0\), must be repeated infinite number of times. So the
constant subsequence \( x_0, x_0, \ldots \) must \((1, 2)^*\alpha^*\) -converges to \( x_0 \).

**Theorem 5.3** Every sequentially \((1, 2)^*\alpha^*\) -compact space is \((1, 2)^*\alpha^*\) -countably compact.

**Proof:** Let \((X, \tau_1, \tau_2)\) be sequentially \((1, 2)^*\alpha^*\) -compact. Since \(X\) is not \((1, 2)^*\alpha^*\) -countably compact .Then there exists countable \((1, 2)^*\alpha^*\) -open cover \( \{U_n\}_{n \in \mathbb{Z}^+} \) which has no finite subcover. Then \( X = \bigcup_{n \in \mathbb{Z}^+} U_n \). Choose

\( x_i \in U_1, x_2 \in U_2 - U_1, x_3 \in U_3 - \bigcup_{i=1,2} U_i, \ldots, x_j \in U_j \) for all \( k \). By our choice of \( \{x_n\} \), there exists \( N \) such that \( x_k \notin U_n \) for all greater than \( k \). Hence there is no subsequence of \( \{x_n\} \) which can \((1, 2)^*\alpha^*\) converge to \( x \). Since \( x \) is arbitrary, the sequence \( \{x_n\} \) has no convergent subsequence which is a contradiction. Therefore \( X \) is \((1, 2)^*\alpha^*\) -countably compact.

**Theorem 5.4:** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a function . then

1) \( f \) is \((1, 2)^*\alpha^*\) -resolute, bijection and \( Y \) is sequentially \((1, 2)^*\alpha^*\) -compact \( \Rightarrow \) \( X \) is sequentially \((1, 2)^*\alpha^*\) -compact.

2) \( f \) is onto, \((1, 2)^*\alpha^*\) -irresolute and \( X \) is sequentially \((1, 2)^*\alpha^*\) -compact \( \Rightarrow \) \( Y \) is sequentially \((1, 2)^*\alpha^*\) -compact.

3) \( f \) is onto \((1, 2)^*\alpha^*\) -irresolute and \( X \) is sequentially \((1, 2)^*\alpha^*\) -compact \( \Rightarrow \) \( Y \) is sequentially \((1, 2)^*\alpha^*\) -compact.

4) \( f \) is onto \((1, 2)^*\alpha^*\) -continuous and \( X \) is sequentially \((1, 2)^*\alpha^*\) -compact \( \Rightarrow \) \( Y \) is sequentially \((1, 2)^*\alpha^*\) -compact.

5) \( f \) is onto, strongly \((1, 2)^*\alpha^*\) -continuous and \( X \) is sequentially \((1, 2)^*\alpha^*\) -compact \( \Rightarrow \) \( Y \) is sequentially \((1, 2)^*\alpha^*\) -compact.

**Proof:** (1) Let \( \{x_n\} \) be a sequence in \( X \). Then \( \{f(x_n)\} \) is a sequence in \( Y \). It has a \((1, 2)^*\alpha^*\) -convergent subsequence \( \{f(x_{n_k})\} \) such that \( f(x_{n_k}) \rightarrow y_0 \) in \( Y \). Then there exists \( x_0 \in X \) such that \( f(x_0) = y_0 \). Let \( U \) be a \((1, 2)^*\alpha^*\) -open set containing \( x_0 \). Then \( f(U) \) is a \((1, 2)^*\alpha^*\) -open set containing \( y_0 \). Then there exists \( N \) such that \( f(x_{n_k}) \in f(U) \) for all \( k \geq N \). Therefore \( f^{-1} \circ f(x_{n_k}) = f^{-1} \circ f(U) \).

Therefore \( x_{n_k} \in U \) for all \( k \geq N \). This proves that \( X \) is sequentially \((1, 2)^*\alpha^*\) -compact. Proof for (2) to (5) is similar to the above.

**REFERENCES**


[6] [1].
