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Some New Subclasses Of Bi-Univalent Functions Defined By Convolution Associated With Linear Differential Operator

M.Thirucheran[1], T. Stalin[2]
Associate Professor and Head, Post Graduate and Research Department of Mathematics[1], Assistant Professor, Department of Mathematics[2],
LN Government College, University of Madras, Chennai, India[1].
Gojan School of Business and Technology, Anna University, Chennai, India[2].
Email: drthirucheran@gmail.com[1], goldstaleen@gmail.com[2]

Abstract: The main object of this paper is investigating a new subclass of bi-univalent function in the open unit disk U which is defined by convolution of Al-Oboudi Differential Operator. And obtained the initial two Taylor -McLaurin co-efficient $|a_2|$ and $|a_3|$ for the subclass $S_{\Sigma,r}^{m,n,b,\delta}$ of Bi-Univalent function.
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1. INTRODUCTION AND DEFINITIONS

Definition 1.1

Let A denote the class of functions f normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \qquad (1.1)$$

Which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$.

For $f \in A$, Al-Oboudi [1] introduces the following operator.

$$D^0 f(z) = f(z),$$
$$D' f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0$$
$$D^n f(z) = D_\delta(D^{n-1} f(z)), \qquad n \in N = 1,2,3,....$$
$$\therefore D^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j , \quad n \in N_0 = N \cup \{0\}. \qquad (1.2)$$

If $\delta = 1$, then we get Salagean [7] differential operator.

Koebe One -Quarter Theorem [5]

The range of every function of class A contains the disk of radius $\left\{ w : |w| < \frac{1}{4} \right\}$.

It is well known that every function $f \in A$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$, $\left(|w| < r_0(f);\, r_0(f) \geq \frac{1}{4} \right)$.

For this inverse function f^{-1}, we have:

$$g(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \qquad (1.3).$$

Let $r(z) = z + \sum_{n=2}^{\infty} r_j z^j$, $(r_j > 0)$ & $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in S_{\Sigma,r}^{m,n,b,\delta}$

Then the Hadamard product $(f * r)$ defined by if and only if $(f * r)(z) = z + \sum_{j=2}^{\infty} r_j a_j z^j \in S_{\Sigma,r}^{m,n,b,\delta}$

Definition 1.2

If both the function f and its inverse function f^{-1} are univalent in U, then the function f is called bi-univalent**.**

For example, $\frac{z}{1-z}, -\log(1-z), \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$, and so on.

However, the familiar Koebe function is not a bi-univalent function.

For example, $z - \frac{z^2}{2}, \frac{z}{1-z^2}$, and so on.

Let the class Σ of bi-univalent function first investigated by Levin [8] and found that

Afterward, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$.

Later**,** Brannan and Taha [3] introduced the new subclass of bi-univalent function of the class Σ like the familiar subclasses $S^*(\alpha)$ and $C(\alpha)$ of starlike and convex functions of $\alpha. (0 \leq \alpha < 1)$,respectively. If a function $f \in A$ is in the class $S_\Sigma^*(\alpha)$ of strongly bi-starlike function of order $\alpha. (0 \leq \alpha < 1)$ if each of the following conditions satisfied $f(z) \in \Sigma$ and $\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}$, $(z \in U)$ and $\left|\arg\left(\frac{wg(z)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}$, $(w \in U)$

where the function g is the extension of f^{-1} to U. Similarly, a function $f \in A$ is in the class $C_{\Sigma}(\alpha)$ of strongly bi-convex function of order α. $(0 \le \alpha < 1)$ if each of the following conditions satisfied $f(z) \in \Sigma$ and $\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}$, $(z \in U)$ and $\left|\arg\left(1+\frac{wg''(z)}{g'(w)}\right)\right| < \frac{\alpha\pi}{2}$, $(w \in U)$.

Where the function g is the extension of f^{-1} to U. For each function $S^*_{\Sigma}(\alpha)$ and $C_{\Sigma}(\alpha)$. They found non-sharp estimates on the first two Taylor - McLaurin co-efficient $|a_2|$ and $|a_3|$.

Recently, Qing-Hua Xu, Ying-Chun Gui and H.M. Srivastava [9], B.A. Frasin and M.K. Aouf [6], Seker.B [11], and R.M. El-Ashwah [10] investigated some subclasses of bi-univalent function and obtained non-sharp estimates on the first two co-efficient.

By motivated this study we introduce and obtained the initial two co-efficient $|a_2|$ and $|a_3|$ for the subclass $S^{m,n,b,\delta}_{\Sigma,r}$.

Definition: 1.3

A function $f(z)$ given by (1.1) is said to be in the class $f \in S^{m,n,b,\delta}_{\Sigma,r}(\alpha)$, $(m,n \in N_0, m > n,\ 0 < \alpha \le 1)$. if the following conditions are satisfied: $f(z), r(z) \in A$ and

$$\left|\arg\left(1+\frac{1}{b}\left(\frac{D^m(f*r)(z)}{D^n(f*r)(z)}-1\right)\right)\right| < \frac{\alpha\pi}{2}, \quad (z \in U) \tag{1.4}$$

and

$$\left|\arg\left(1+\frac{1}{b}\left(\frac{D^m(g*r)(w)}{D^n(g*r)(w)}-1\right)\right)\right| < \frac{\alpha\pi}{2}, \quad (w \in U) \tag{1.5}$$

Where the function g is given by (1.3).

Definition: 1.4

A function $f(z)$ given by (1.1) is said to be in the class $f \in S^{m,n,b,\delta}_{\Sigma,r}(\gamma)$, $(m,n \in N_0, m > n,\ 0 < \alpha \le 1)$, if the following conditions are satisfied: $f(z), r(z) \in A$ and

$$\Re\left(1+\frac{1}{b}\left(\frac{D^m(f*r)(z)}{D^n(f*r)(z)}-1\right)\right) > \gamma, (z \in U) \tag{1.6}$$

$$\& \Re\left(1+\frac{1}{b}\left(\frac{D^m(g*r)(w)}{D^n(g*r)}-1\right)\right) > \gamma, (w \in U) \tag{1.7}$$

Where the function g is given by (1.3).

Definition: 1.5

A function $h(z), p(z): U \to C$ satisfy the conditions $\min\{\Re(h(z)), \Re(p(z))\} > 0$, $z \in U$ and $h(0) = p(0) = 1$.

For a function $f \in S^{m,n,b,\delta}_{\Sigma,r}(h(z), p(z))$ defined by (1.1), $(m,n \in N_0, m > n,\ 0 < \alpha \le 1)$. if the following conditions are satisfied: $f, r \in A$ and

$$\left(1+\frac{1}{b}\left(\frac{D^m(f*r)(z)}{D^n(f*r)(z)}-1\right)\right) \in h(z), (z \in U) \tag{1.8}$$

$$\& \left(1+\frac{1}{b}\left(\frac{D^m(g*r)(w)}{D^n(g*r)(w)}-1\right)\right) \in p(z), (w \in U)$$

(1.9)

Where the function g is given by (1.3).

Remarks

(i). $S^{m,n,1,1}_{\Sigma,1}(\alpha) = H^{m,n}_{\Sigma}(\alpha)$,(Seker.B [11])

(ii). $S^{1,0,1,1}_{\Sigma,1}(\alpha) = S^*_{\Sigma}(\alpha)$, (Brannan and Taha [3])

(iii). $S^{2,1,1,1}_{\Sigma,1}(\alpha) = C_{\Sigma}(\alpha)$, (Brannan and Taha [3])

2. MAIN RESULT

To derive our main results, we should recall the following lemma [4].

Lemma 2.1

Let $h \in P$ the family of all functions h analytic in U for which $\mathrm{Re}\{h(z) > 0\}$ and have the form $h(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$ for $z \in U$. Then $|p_n \le 2|$, for each n.

Theorem 2.2

Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S^{m,n,b,\delta}_{\Sigma}(h, p)$. Then

$$|a_2| \le \frac{2\sqrt{|b|}}{r_2\sqrt{2\left[\begin{array}{l}\left((1+2\delta)^m-(1+2\delta)^n\right)\\ -\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{array}\right]}}$$

(2.1)

and

$$|a_3| \le \frac{1}{r_3}\left(\frac{4|b|^2}{\left[(1+\delta)^m-(1+\delta)^n\right]^2} + \frac{2|b|}{\left[(1+2\delta)^m-(1+2\delta)^n\right]}\right) \tag{2.2}$$

Proof

Let consider the function h and p Satisfying the conditions of the definition (1.5) with the form $h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \ldots\ldots$, $z \in U$ & $p(w) = 1 + p_1 w + p_2 w^2 + p_3 w^3 + \ldots\ldots$, $w \in U$ respectively.

Since $f, r \in S_{\Sigma,r}^{m,n,b,\delta}(h,p)$,then

$$\left(1+\frac{1}{b}\left(\frac{D^m(f*r)(z)}{D^n(f*r)(z)}-1\right)\right)=h(z),\ (z\in U) \quad (2.3)$$

and

$$\left(1+\frac{1}{b}\left(\frac{D^m(g*r)(w)}{D^n(g*r)(w)}-1\right)\right)=p(z),\ (w\in U) \quad (2.4)$$

respectively.

By equating the coefficient of (2.3 and (2.4), we get

$$\left((1+\delta)^m-(1+\delta)^n\right)a_2r_2=bh_1 \quad (2.5)$$

$$\left((1+2\delta)^m-(1+2\delta)^n\right)a_3r_3 - \left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)a_2^2r_2^2=bh_2 \quad (2.6)$$

$$-\left((1+\delta)^m-(1+\delta)^n\right)a_2r_2=bp_1 \quad (2.7)$$

and

$$-\left((1+2\delta)^m-(1+2\delta)^n\right)a_3r_3 + \left[\begin{matrix}2\left((1+2\delta)^m-(1+2\delta)^n\right)\\ -\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{matrix}\right]a_2^2r_2^2=bp_2 \quad (2.8)$$

From (2.5)&(2.7),

$$h_1=-p_1 \quad (2.9)$$

And

$$2\left[(1+\delta)^m-(1+\delta)^n\right]^2a_2^2r_2^2=b^2(h_1^2+p_1^2) \quad (2.10)$$

$$\therefore\ a_2^2=\frac{b^2(h_1^2+p_1^2)}{2r_2^2\left((1+\delta)^m-(1+\delta)^n\right)^2} \quad (2.11)$$

From (2.6)&(2.8),

$$2\left[\begin{matrix}\left((1+2\delta)^m-(1+2\delta)^n\right)\\ -\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{matrix}\right]a_2^2r_2^2=b(h_2+p_2) \quad (2.12)$$

$$\therefore\ a_2^2=\frac{b(h_2+p_2)}{2r_2^2\left[\begin{matrix}\left((1+2\delta)^m-(1+2\delta)^n\right)\\ -\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{matrix}\right]} \quad (2.13)$$

$$2\left((1+2\delta)^m-(1+2\delta)^n\right)a_3r_3 = 2\left((1+2\delta)^m-(1+2\delta)^n\right)a_2^2r_2^2+b(h_2-p_2) \quad (2.14)$$

$$\therefore a_3=\frac{1}{r_3}\left(\frac{b^2(h_1^2+p_1^2)}{2\left((1+\delta)^m-(1+\delta)^n\right)^2}+\frac{b(h_2-p_2)}{2\left((1+2\delta)^m-(1+2\delta)^n\right)}\right) \quad (2.15)$$

From Lemma (2.1),(2.11),(2.13) and (2.15), we get

$$|a_2|\le\frac{2\sqrt{|b|}}{r_2\sqrt{2\left[\begin{matrix}\left((1+2\delta)^m-(1+2\delta)^n\right)\\ -\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{matrix}\right]}}$$

and
$$|a_3|\le\frac{1}{r_3}\left(\frac{4|b|^2}{\left[(1+\delta)^m-(1+\delta)^n\right]^2}+\frac{2|b|}{\left[(1+2\delta)^m-(1+2\delta)^n\right]}\right)$$

This completes the theorem (2.2).

Theorem 2.3

Let the function f, r given by (1.1) be in the bi-univalent function class $S_{\Sigma,r}^{m,n,b,\delta}(\alpha)$ .Then

$$|a_2|\le\frac{2\alpha\sqrt{|b|}}{r_2\sqrt{2\alpha\left[\begin{matrix}\left((1+2\delta)^m-(1+2\delta)^n\right)\\ -\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{matrix}\right]}} \quad (2.16)$$

and
$$|a_3|\le\frac{1}{r_3}\left(\frac{4|b|^2\alpha^2}{\left[(1+\delta)^m-(1+\delta)^n\right]^2}+\frac{2|b|\ \alpha}{\left[(1+2\delta)^m-(1+2\delta)^n\right]}\right)$$

(2.17)

Proof

Let consider the function h and p Satisfying the conditions of the definition (1.5) with the form

$$h(z)=1+h_1z+h_2z^2+h_3z^3+........,\ z\in U$$

and

$$p(w)=1+p_1w+p_2w^2+p_3w^3+........,\ w\in U$$

respectively.

Since $f, r \in S_{\Sigma,r}^{m,n,b,\delta}(\alpha)$,then

$$\left(1+\frac{1}{b}\left(\frac{D^m(f*r)(z)}{D^n(f*r)(z)}-1\right)\right)=\left[h(z)\right]^\alpha,\ (z\in U) \quad (2.18)$$

and
$$\left(1+\frac{1}{b}\left(\frac{D^m(g*r)(w)}{D^n(g*r)(w)}-1\right)\right)=\left[p(z)\right]^\alpha,\ (w\in U) \quad (2.19)$$

respectively.

From Lemma (2.1),(2.11),(2.13) and (2.15), we get

$$|a_2|\le\frac{2\alpha\sqrt{|b|}}{r_2\sqrt{2\alpha\left[\begin{matrix}\left((1+2\delta)^m-(1+2\delta)^n\right)\\ -\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{matrix}\right]}}$$

and

$$|a_3|\le\frac{1}{r_3}\left(\frac{4|b|^2\alpha^2}{\left[(1+\delta)^m-(1+\delta)^n\right]^2}+\frac{2|b|\ \alpha}{\left[(1+2\delta)^m-(1+2\delta)^n\right]}\right)$$

This completes the theorem (2.3).

Theorem 2.4

Let the function f, r given by (1.1) be in the bi-univalent function class $S_{\Sigma,r}^{m,n,b,\delta}(\gamma)$ .Then

$$|a_2| \le \frac{2(1-\gamma)\sqrt{|b|}}{r_2\sqrt{2(1-\gamma)\left[\begin{array}{l}\left((1+2\delta)^m-(1+2\delta)^n\right)\\-\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{array}\right]}} \quad (2.20)$$

$$\text{and } |a_3| \le \frac{1}{r_3}\left(\frac{4|b|^2(1-\gamma)^2}{\left[(1+\delta)^m-(1+\delta)^n\right]^2}+\frac{2|b|(1-\gamma)}{\left[(1+2\delta)^m-(1+2\delta)^n\right]}\right) \quad (2.21)$$

Proof

Let consider the function h and p Satisfying the conditions of the definition (1.5) with the form

$h(z)=1+h_1z+h_2z^2+h_3z^3+........, z\in U$ and

$p(w)=1+p_1w+p_2w^2+p_3w^3+........, \ w\in U$

respectively.

Since $f, r\in S_{\Sigma,r}^{m,n,b,\delta}(\gamma)$,then

$$\left(1+\frac{1}{b}\left(\frac{D^m(f*r)(z)}{D^n(f*r)(z)}-1\right)\right)=\gamma+(1-\gamma)h(z),$$

$(z\in U)$ (2.22)

$$\text{and } \left(1+\frac{1}{b}\left(\frac{D^m(g*r)(w)}{D^n(g*r)(w)}-1\right)\right)=\gamma+(1-\gamma)p(z)$$

, $(w\in U)$ (2.23)

respectively.

From Lemma (2.1),(2.11),(2.13) and (2.15), we get

$$|a_2| \le \frac{2(1-\gamma)\sqrt{|b|}}{r_2\sqrt{2(1-\gamma)\left[\begin{array}{l}\left((1+2\delta)^m-(1+2\delta)^n\right)\\-\left((1+\delta)^{m+n}-(1+\delta)^{2n}\right)\end{array}\right]}}$$

$$\text{and } |a_3| \le \frac{1}{r_3}\left(\frac{4|b|^2(1-\gamma)^2}{\left[(1+\delta)^m-(1+\delta)^n\right]^2}+\frac{2|b|(1-\gamma)}{\left[(1+2\delta)^m-(1+2\delta)^n\right]}\right)$$

Letting $\delta=1$ in theorem (2.2), (2.3) and (2.4), we obtain the following corollaries.

Corollary 2.5

Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_{\Sigma,1}^{m,n,b,1}(h,p)$

.Then $|a_2| \le \sqrt{\frac{2|b|}{\left[\left((3)^m-(3)^n\right)-\left((2)^{m+n}-(2)^{2n}\right)\right]}}$

(2.24)

$$\text{and } |a_3| \le \frac{4|b|^2}{\left[(2)^m-(2)^n\right]^2}+\frac{2|b|}{\left[(3)^m-(3)^n\right]}$$

(2.25)

Corollary 2.6

Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_{\Sigma,1}^{m,n,b,1}(\alpha)$ .Then (using definition (1.3))

$$|a_2| \le \sqrt{\frac{2|b|\alpha}{\left[\left((3)^m-(3)^n\right)-\left((2)^{m+n}-(2)^{2n}\right)\right]}} \quad (2.26)$$

$$\text{and } |a_3| \le \frac{4|b|^2\alpha^2}{\left[(2)^m-(2)^n\right]^2}+\frac{2|b|\alpha}{\left[(3)^m-(3)^n\right]}$$

(2.27)

Corollary 2.7

Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_{\Sigma,1}^{m,n,b,1}(\gamma)$ .Then (using definition (1.4))

$$|a_2| \le \sqrt{\frac{2|b|(1-\gamma)}{\left[\left((3)^m-(3)^n\right)-\left((2)^{m+n}-(2)^{2n}\right)\right]}}$$

(2.28)

$$\text{and } |a_3| \le \frac{4|b|^2(1-\gamma)^2}{\left[(2)^m-(2)^n\right]^2}+\frac{2|b|(1-\gamma)}{\left[(3)^m-(3)^n\right]}$$

(2.29)

Letting $\delta=1\ \&\ b=1$ in theorem (2.2), (2.3) and (2.4), we obtain the following corollaries Seker [6].

Corollary 2.8

Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_{\Sigma,1}^{m,n,1,1}(h,p)$

.Then $|a_2| \le \sqrt{\frac{2}{\left[\left((3)^m-(3)^n\right)-\left((2)^{m+n}-(2)^{2n}\right)\right]}}$

(2.30)

$$\text{and } |a_3| \le \frac{4}{\left[(2)^m-(2)^n\right]^2}+\frac{2}{\left[(3)^m-(3)^n\right]} \quad (2.31)$$

Corollary 2.9

Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_{\Sigma,1}^{m,n,1,1}(\alpha)$ .Then (using definition (1.3))

$$|a_2| \le \sqrt{\frac{2\alpha}{\left[\left((3)^m-(3)^n\right)-\left((2)^{m+n}-(2)^{2n}\right)\right]}} \quad (2.32)$$

$$\text{and } |a_3| \le \frac{4\alpha^2}{\left[(2)^m-(2)^n\right]^2}+\frac{2\alpha}{\left[(3)^m-(3)^n\right]} \quad (2.33)$$

Corollary 2.10

Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_{\Sigma,1}^{m,n,1,1}(\gamma)$ .Then (using definition (1.4))

$$|a_2| \le \sqrt{\frac{2(1-\gamma)}{\left[\left((3)^m-(3)^n\right)-\left((2)^{m+n}-(2)^{2n}\right)\right]}} \quad (2.34)$$

$$\text{and } |a_3| \le \frac{4(1-\gamma)^2}{\left[(2)^m-(2)^n\right]^2}+\frac{2(1-\gamma)}{\left[(3)^m-(3)^n\right]} \quad (2.35)$$

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