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Heat Equation Generated By Two Dimensional l-Difference Operator

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Abstract: The heat equation is of fundamental importance in diverse scientific fields. Heat is a form of energy that exists in any material. For example, the temperature in an object changes with time and the position within the object. In this paper Generalized Heat equation generated by two dimensional $\mathfrak{l}$ -difference operator.
Keywords: Partial difference equation, partial difference operator, discrete heat equation

1. INTRODUCTION

In 1984, Jerzy Popenda [6] introduced the difference operator Δ_α defined on u(k) as $\Delta_\alpha u(k)=u(k+1)-\alpha u(k)$. In 1989, Miller and Rose [5] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the inverse fractional difference operator ([2],[3]). Several formula on higher order partial sums on arithmetic, geometric progressions and products of n-consecutive terms of arithmetic progression have been derived in [7]

In 2011, M.Maria Susai Manuel, et.al, [4] extended the operator Δ_α to generalized α−difference operator as $\Delta_{\alpha(l)}v(k) = v(k + l) - \alpha v(k)$ for the real valued function v(k).The generalized difference operator with n-shift values $\mathfrak{l}=(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\cdots,\mathfrak{l}_n)\neq 0$ on a real valued function v(k): $\mathbb{R}^n\to\mathbb{R}$ is defined as

$$\Delta_{(\mathfrak{l})} = v(k_1+\mathfrak{l}_1, k_2+\mathfrak{l}_2, \dots\dots, k_n+\mathfrak{l}_n) - v(k_1,k_2,\dots\dots,k_2)$$

A linear generalized partial difference equation is of the form $\Delta_{(l)}v(k)=u(k)$, then the inverse of generalized partial difference equation

$v(k)=\Delta^{-1} u(k)$, (1.1)

u(k): $\mathbb{R}^n\to\mathbb{R}$ is given f unction, A function v(k): $\mathbb{R}^n\to\mathbb{R}$ satisfying (1.1) is called a solution of Equation (1.1). Equation (1.1) has a numerical solution of the form

$v(k)-v(k-m\,\mathfrak{l})= \left(\sum_{r=1}^{m} u(k-r\,\mathfrak{l})\right)$ (1.2)

where $k-r\mathfrak{l} =(k_1-r\,\mathfrak{l}_1, k_2-r\,\mathfrak{l}_2 \cdots, k_n-r\,\mathfrak{l}_n)$, m is any positive integer. Relation **(**1.2) is the basic inverse principle with respect to $\Delta_{(l)}$

Partial difference and differential equations take vital role in Heat equation. To reach partial differential equation from Partial difference equation. We need to introduce an operator called generalized two dimensional difference operators which is defined as

$$\Delta_{(\mathfrak{l}_1,\mathfrak{l}_2)} = v(k_1+\mathfrak{l}_1, k_2+\mathfrak{l}_2) - v(k_1,k_2)$$

For real valued function $v(k_1,k_2)$ the equation

$$f\left(\Delta_{(\mathfrak{l}_1,\mathfrak{l}_2)}, \Delta_{(\mathfrak{l}_1,0)}, \Delta_{(\mathfrak{l}_2,0)}\right)\ v(k_1,k_2) = 0$$

1.1. Generalized Discrete Heat equation

Consider the temperature of a very long rod. Assume that the rod is so long that it can be laid on top of the set $\mathscr{R}$ of real numbers. Let $v(k_1,k_2)$ be the temperature at the real time k_1 and real position k_2 of the rod. At time k_1 if the temperature $v(k_1,k_2-\mathfrak{l}_2)$, $\mathfrak{l}_2>0$ is higher than $v(k_1,k_2)$, heat will flow from the point $k_2-\mathfrak{l}_2$ to k_2. The amount increase is $v(k_1+\mathfrak{l}_1,k_2)-v(k_1,k_2)$ and it is reasonable to postulate that the increase is proportional to the difference $v(k_1,k_2-\mathfrak{l}_2)-v(k_1,k_2)$ say $\delta v(k_1,k_2-\mathfrak{l}_2)-v(k_1,k_2)$ Where δ is a positive diffusion rate constant.

$v(k_1+\mathfrak{l}_1,k_2)-v(k_1,k_2) = \delta[v(k_1,k_2-\mathfrak{l}_2)-v(k_1,k_2)], \delta>0$ (1.3)

$$\Delta_{(\mathfrak{l}_1,0)}\, v(k_1,k_2) = \delta\,[\Delta_{(0,-\mathfrak{l}_2)}\ v(k_1,k_2)]$$

$$\delta = \frac{v(k_1+\mathfrak{l}_1,k_2)-v(k_1,k_2)}{v(k_1,k_2-\mathfrak{l}_2)-v(k_1,k_2)}$$

$$= \frac{(k_1+\mathfrak{l}_1+k_2)-(k_1+k_2)}{(k_1+k_2-\mathfrak{l}_2)-(k_1+k_2)} = -\frac{\mathfrak{l}_1}{\mathfrak{l}_2}$$

Substituting δ value in equation (1.3) we get,

$$(k_1+\mathfrak{l}_1+k_2)-(k_1+k_2) = -\frac{\mathfrak{l}_1}{\mathfrak{l}_2}[(k_1+k_2-\mathfrak{l}_2)-(k_1+k_2)]$$

$$-\frac{\mathfrak{l}_1}{\mathfrak{l}_2} = -\frac{\mathfrak{l}_1}{\mathfrak{l}_2}$$

Equation (1.3) is a model of simple heat equation. Where k_1 and k_2 are variables and $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are Parameters, where $\mathfrak{l}_1,\mathfrak{l}_2\geq 2$.

Two dimensional $\mathfrak{l}$ -heat equation with variable coefficient is defined as

$v(k_1+\mathfrak{l}_1,k_2)-v(k_1,k_2) = \delta(k_1,k_2)$
$[v(k_1,k_2-\mathfrak{l}_2)-v(k_1,k_2)], \delta(k_1,k_2)>0$ (1.4)

Where δ is a function of $k_1\ and\ k_2$. The value of $\delta(k_1,k_2) = -\frac{\mathfrak{l}_1}{\mathfrak{l}_2}$

Theorem: 1.1.1

From equation (1.3) we have, since $\Delta^{-1}_{(\mathrm{l}_1,\mathrm{l}_2)}$ *is linear*

$$v(k_1,k_2) = \delta\Delta^{-1}_{(\mathrm{l}_1,0)}[\Delta_{(0,-\mathrm{l}_2)}\, v(k_1,k_2)]$$

Proof: From experimental value, we take $\Delta_{(0,-\mathrm{l}_2)}\, v(k_1,k_2) = u(k_1,k_2)$

(1.5)

$v(k_1,k_2) = \delta\Delta^{-1}_{(\mathrm{l}_1,0)}[\, u(k_1,k_2)]$

Replace k_1 by $k_1 - r\mathrm{l}_1$ in equation (1.5) we get $v(k_1,k_2) = \delta \sum_{r=1}^{m} u(\, k_1 - r\mathrm{l}_1, k_2)$

We have $v(k_1,k_2) - v(k_1 - m\mathrm{l}_1, k_2)$

$= \delta \sum_{r=1}^{m} u(\, k_1 - r\mathrm{l}_1, k_2)$ (1.6)

In equation (1.6) $u(k_1 - r\mathrm{l}_1, k_2)$ is obtained by replacing k_1 by $k_1 - r\mathrm{l}_1$ in (1.5) and Substituting (1.5) in (1.6) we get,

$v(k_1,k_2) - v(k_1 - m\mathrm{l}_1, k_2)$

$= \delta \sum_{r=1}^{m} \Delta_{(0,-\mathrm{l}_2)}\, [v(\, k_1 - r\mathrm{l}_1, k_2)]$

$v(k_1,k_2) - v(k_1 - m\mathrm{l}_1, k_2)$

$= \delta \sum_{r=1}^{m} \, [v(\, k_1 - r\mathrm{l}_1, k_2 - \mathrm{l}_2) - v(k_1 - r\mathrm{l}_1, k_2)]$

(1.7)

Equation (1.7) is a numerical solution of equation (1.3)

Corollary: 1.1.2

From equation (1.4) we have, since $\Delta^{-1}_{(\mathrm{l}_1,\mathrm{l}_2)}$ *is linear*

$$v(k_1,k_2) = \delta(k_1,k_2)\ \Delta^{-1}_{(\mathrm{l}_1,0)}\, [\Delta_{(0,-\mathrm{l}_2)}\, v(k_1,k_2)]$$

Proof: Replace δ by $\delta(k_1,k_2)$ in theorem (1.1.1)

Example: 1.1.3

Take $k_1 = 3, k_2 = 4, \mathrm{l}_1 = 2, \mathrm{l}_2 = 2, \mathrm{m} = 2$, $\delta = -\frac{\mathrm{l}_1}{\mathrm{l}_2} = -1$ in (1.7)

$$(k_1 + k_2) - (k_1 - m\mathrm{l}_1 + k_2)$$

$$= \delta \sum_{r=1}^{m} [(\, k_1 - r\mathrm{l}_1 + k_2 - \mathrm{l}_2) - (k_1 - r\mathrm{l}_1 + k_2)]$$

$$4 = 4$$

Theorem: 1.1.4

Let $\delta \in \Re, k_1, k_2$ *be the variables and* $\mathrm{l}_1, \mathrm{l}_2$ *be the difference operator, then*

$$v(k_1,k_2) = \frac{1}{(1-\delta)^m}\, v(k_1 + m\mathrm{l}_1, k_2) - \sum_{r=1}^{m} \frac{\delta}{(1-\delta)^r}\, v(k_1 + (r-1)\mathrm{l}_1, k_2 - \mathrm{l}_2)$$

Proof:

Consider the two dimensiona l-difference heat equation,

$v(k_1 + \mathrm{l}_1, k_2) - v(k_1,k_2)$

$= \delta[v(k_1, k_2 - \mathrm{l}_2) - v(k_1,k_2)]$

$v(k_1 + \mathrm{l}_1, k_2) - \delta[\, v(k_1, k_2 - \mathrm{l}_2)]$

$= (1-\delta)\, v(k_1,k_2)$

$v(k_1,k_2) = \frac{1}{(1-\delta)}\, v(k_1 + \mathrm{l}_1, k_2)$

$- \frac{\delta}{(1-\delta)}\, v(k_1, k_2 - \mathrm{l}_2)$ (1.8)

Replace k_1 by $k_1 + \mathrm{l}_1$ inequation (1.8) we get,

$v(k_1 + \mathrm{l}_1, k_2) = \frac{1}{(1-\delta)}\, v(k_1 + 2\mathrm{l}_1, k_2)$

$- \frac{\delta}{(1-\delta)}\, v(k_1 + \mathrm{l}_1, k_2 - \mathrm{l}_2)$ (1.9)

Substituting equation (1.9) in (1.8) we get,

$v(k_1,k_2) = \frac{1}{(1-\delta)^2}\, v(k_1 + 2\mathrm{l}_1, k_2)$

$- \frac{\delta}{(1-\delta)^2}\, v(k_1 + \mathrm{l}_1, k_2 - \mathrm{l}_2)$

$- \frac{\delta}{(1-\delta)}\, v(k_1, k_2 - \mathrm{l}_2)$ (1.10)

Replace k_1 by $k_1 + 2\mathrm{l}_1$ in equation (1.8) we get,

$v(k_1 + 2\mathrm{l}_1, k_2) = \frac{1}{(1-\delta)}\, v(k_1 + 3\mathrm{l}_1, k_2)$

$- \frac{\delta}{(1-\delta)}\, v(k_1 + 2\mathrm{l}_1, k_2 - \mathrm{l}_2)$ (1.11)

Substituting equation (1.11) in (1.10) we get,

$$v(k_1,k_2) = \frac{1}{(1-\delta)^3}\, v(k_1 + 3\mathrm{l}_1, k_2) - \frac{\delta}{(1-\delta)^3}\, v(k_1 + 2\mathrm{l}_1, k_2 - \mathrm{l}_2) - \frac{\delta}{(1-\delta)^2}\, v(k_1 + \mathrm{l}_1, k_2 - \mathrm{l}_2)$$

$- \frac{\delta}{(1-\delta)}\, v(k_1, k_2 - \mathrm{l}_2)$ (1.12)

In General,

$v(k_1,k_2) = \frac{1}{(1-\delta)^m}\, v(k_1 + m\mathrm{l}_1, k_2)$

$- \sum_{r=1}^{m} \frac{\delta}{(1-\delta)^r}\, v(k_1 + (r-1)\mathrm{l}_1, k_2 - \mathrm{l}_2)$ (1.13)

Corollary: 1.1.5

Let $\delta \in \Re, k_1, k_2$ *be the variables and* $\mathrm{l}_1, \mathrm{l}_2$ *be the difference operator, then*

$$v(k_1,k_2) = \frac{1}{\prod_{r=1}^{m}(1 - \delta\,(k_1 + (r-1)\mathrm{l}_1, k_2))}\, v(k_1 + m\mathrm{l}_1, k_2) - \sum_{r=1}^{m} \frac{\delta(k_1 + (r-1)\mathrm{l}_1, k_2)}{\prod_{s=1}^{r}(1 - \delta(\, k_1 + (s-1)\mathrm{l}_1, k_2))}\, v(k_1 + (r-1)\mathrm{l}_1, k_2 - \mathrm{l}_2)$$

Proof: Replace δ by $\delta(k_1,k_2)$ in theorem (1.1.4)

Example: 1.1.6

Take $k_1 = 3, k_2 = 4, \mathrm{l}_1 = 2, \mathrm{l}_2 = 2, \mathrm{m} = 2$, $\delta = -\frac{\mathrm{l}_1}{\mathrm{l}_2} = -1$ in (1.13)

$(k_1 + k_2) = \frac{1}{(1-\delta)^m}(k_1 + m\mathrm{l}_1 + k_2)$

$- \sum_{r=1}^{m} \left[\frac{\delta}{(1-\delta)^r}(k_1 + (r-1)\mathrm{l}_1 + k_2 - \mathrm{l}_2)\right]$

3+4 = 2.75 + 4.25

7 = 7

Theorem: 1.1.7

Let $\delta \in \Re, k_1, k_2$ *be the variables and* l_1, l_2 *be the difference operator, then*

$$v(k_1,k_2) = \frac{1}{(1-\delta)}\, v(k_1 + l_1, k_2) - \frac{\delta}{(1-\delta)^{(m+1)}} v(k_1 + m l_1, k_2 - l_2) + \sum_{r=1}^{m} \frac{\delta^2}{(1-\delta)^{(r+1)}}\, v(k_1 + (r-1) l_1, k_2 - 2l_2)$$

Proof:

Consider the two dimensiona l-difference heat equation (1.8), we have

$$v(k_1,k_2) = \frac{1}{(1-\delta)}\, v(k_1 + l_1, k_2) - \frac{\delta}{(1-\delta)}\, v(k_1, k_2 - l_2)$$

Replace k_2 by $k_2 - l_2$ in equation (1.8) we get,

$$v(k_1,k_2 - l_2) = \frac{1}{(1-\delta)}\, v(k_1 + l_1, k_2 - l_2) - \frac{\delta}{(1-\delta)}\, v(k_1, k_2 - 2l_2) \qquad (1.14)$$

Substituting equation (1.14) in (1.8) we get,

$$v(k_1,k_2) = \frac{1}{(1-\delta)}\, v(k_1 + l_1, k_2) - \frac{\delta}{(1-\delta)^2}\, v(k_1 + l_1, k_2 - l_2) + \frac{\delta^2}{(1-\delta)^2}\, v(k_1, k_2 - 2l_2) \ (1.15)$$

Replace (k_1, k_2) by $(k_1 + l_1, k_2 - l_2)$ in equation (1.8) we get,

$$v(k_1 + l_1, k_2 - l_2) = \frac{1}{(1-\delta)}\, v(k_1 + 2l_1, k_2 - l_2) - \frac{\delta}{(1-\delta)}\, v(k_1 + l_1, k_2 - 2l_2) \qquad (1.16)$$

Substituting equation (1.16) in (1.15) we get,

$$v(k_1,k_2) = \frac{1}{(1-\delta)}\, v(k_1 + l_1, k_2) - \frac{\delta}{(1-\delta)^3}\, v(k_1 + 2l_1, k_2 - l_2) + \frac{\delta^2}{(1-\delta)^3}\, v(k_1 + l_1, k_2 - 2l_2) + \frac{\delta^2}{(1-\delta)^2}\, v(k_1, k_2 - 2l_2) \qquad (1.17)$$

Replace (k_1, k_2) by $(k_1 + 2l_1, k_2 - l_2)$ in equation (1.8) we get,

$$v(k_1 + 2l_1, k_2 - l_2) = \frac{1}{(1-\delta)}\, v(k_1 + 3l_1, k_2 - l_2) - \frac{\delta}{(1-\delta)}\, v(k_1 + 2l_1, k_2 - 2l_2) \qquad (1.18)$$

Substituting equation (1.18) in (1.17) we get,

$$v(k_1,k_2) = \frac{1}{(1-\delta)}\, v(k_1 + l_1, k_2) - \frac{\delta}{(1-\delta)^4} v(k_1 + 3l_1, k_2 - l_2) + \frac{\delta^2}{(1-\delta)^4} v(k_1 + 2l_1, k_2 - 2l_2) + \frac{\delta^2}{(1-\delta)^3}\, v(k_1 + l_1, k_2 - 2l_2) + \frac{\delta^2}{(1-\delta)^2}\, v(k_1, k_2 - 2l_2)$$

(1.19) In General,

$$v(k_1,k_2) = \frac{1}{(1-\delta)}\, v(k_1 + l_1, k_2) - \frac{\delta}{(1-\delta)^{(m+1)}} v(k_1 + m l_1, k_2 - l_2) + \sum_{r=1}^{m} \frac{\delta^2}{(1-\delta)^{(r+1)}}\, v(k_1 + (r-1) l_1, k_2 - 2l_2)$$

(1.20)

Corollary: 1.1.8

Let $\delta \in \Re, k_1, k_2$ *be the variables and* l_1, l_2 *be the difference operator, then*

$$v(k_1,k_2) = \frac{1}{(1-\delta(k_1,k_2))} v(k_1 + l_1, k_2) - \frac{\delta(k_1,k_2)}{(1-\delta(k_1,k_2)) \prod_{r=1}^{m} (1 - \delta(k_1 + (r-1) l_1, k_2 - l_2)} v(k_1 + m l_1, k_2 - l_2) + \sum_{r=1}^{m} \frac{\delta(k_1,k_2)\delta(k_1 + (r-1)l_1, k_2 - l_2)}{(1-\delta(k_1,k_2)) \prod_{s=1}^{r} (1 - \delta(k_1 + (s-1) l_1, k_2 - l_2)} v(k_1 + (r-1) l_1, k_2 - 2l_2)$$

Proof: Replace δ by $\delta(k_1, k_2)$ in theorem (1.1.7)

Example: 1.1.9

Take $k_1 = 3, k_2 = 4, l_1 = 2, l_2 = 2, \mathrm{m} = 2$, $\delta = -\frac{l_1}{l_2} = -1$ in (1.20)

$$(k_1 + k_2) = \frac{1}{(1-\delta)} (k_1 + l_1 + k_2) - \frac{\delta}{(1-\delta)^{(m+1)}} (k_1 + m l_1 + k_2 - l_2) + \sum_{r=1}^{m} \frac{\delta^2}{(1-\delta)^{(r+1)}} (k_1 + (r-1) l_1 + k_2 - 2l_2)$$

$$3 + 4 = \frac{1}{1+1} (3 + 2 + 4) - \frac{(-1)}{(1+1)^3} (3 + 4 + 4 - 2) + [\frac{(-1)^2}{(1+1)^2} (3 + 0 + 4 - 4) + \frac{(-1)^2}{(1+1)^3} (3 + 2 + 4 - 4)]$$

$$3 + 4 = 4.5 + 1.125 + 0.75 + 0.625$$

$$7 = 7$$

2. CONCLUSION:

This equation can be applied in solving the heat flow that is related in science and engineering. The accuracy in using difference method is more reliable rather than using other method. In this paper Generalized Heat equation generated by two dimensional l -difference operator.

REFERENCES

[1] Agarwal R.P., Difference Equations and Inequalities, Marcel Dekker, New York,2000.

[2] Bastos, N.R.O., Ferreira, R.A.C. and Torres, D.F.M. (2011) Discrete-Time Fractional

Variational Problems. Signal Processing, 91, 513-524

[3] Ferreira, R.A.C. and Torres, D.F.M. (2011) Fractional h-Difference Equations Arising from the Calculus of Variations. Applicable Analysis and Discrete Mathematics, 5, 110-121.

[4] Maria Susai Manuel, M., Chandrasekar, V. and Britto Antony Xavier, G. (2011) Solutions and Applications of Certain Class of α-Difference Equations. International Journal of Applied Mathematics, 24, 943-954

[5] Miller, K.S and Ross, B. (1989) Fractional Difference Calculus in Univalent Functions. Horwood, Chichester, 139-152.

[6] Popenda, J. and Szmanda, B. (1984) On the Oscillation of Solutions of Certain Difference Equations. DemonstratioMathematica, XVII, 153-164.

[7] Susai Manuel, M., Britto Antony Xavier, G., Chandrasekar, V. and Pugalarasu, R. (2012) Theory and application of the Generalized Difference Operator of the nth kind (Part I). Demonstratio Mathematica, 45, 95-106