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ON SCHUR M - POWER CONVEXITY OF THE GENERALIZED GEOMETRIC BONFERRONI MEAN

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ABSTRACT. : In this paper, we discuss the Schur m -power convexity of the generalized geometric Bonferroni mean by introducing three non-negative parameters p, q, r under the condition of $p + q + r \neq 0$, is discussed.

1. Introduction

The Schur convexity of functions relating to special means is a very significant research subject and has attracted the interest of many mathematicians. There are numerous articles written on this topic in recent years; see [1],[2] and the references therein. As supplements to the Schur convexity of functions, the Schur geometrically convex functions and Schur harmonically convex functions were investigated by Zhang and Yang [3], Chu, Zhang and Wang [4], Chu and Xia [5], Chu, Wang and Zhang [6], Shi and Zhang [7, 8], Meng, Chu and Tang [9], Zheng, Zhang and Zhang [10]. These properties of functions have been found to be useful in discovering and proving the inequalities for special means (see [11-14]).

Recently, it has come to our attention that a type of means which is symmetrical on n variables $x_1, x_2, ..., x_n$ and involves two parameters, it was initially proposed by Bonferroni [15], as follows:

$$B^{p,q}(X) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} x_i^p x_j^q\right)^{\frac{1}{p+q}} \tag{1.1}$$

where $X = (x_1, x_2, ..., x_n)$, $x_i \geq 0$, $i = 1, 2, \ . \ . \ . \ , n$, $p, q \geq 0$ and $p + q \neq 0$. $B^{p,q}(x)$ is called the Bonferroni mean. It has important application in multi criteria decision-making (see [16-21]).

Beliakov, James and Mordelov et al. [22] generalized the Bonferroni mean by introducing three parameters $p, q, r, i.e.,$

$$B^{p,q,r}(X) = \left(\frac{1}{n(n-1)(n-2)} \sum_{i,j,k=1, i\neq j\neq k}^{n} x_i^p x_j^q x_k^r\right)^{\frac{1}{p+q+r}} \tag{1.2}$$

where $X = (x_1, x_2, ..., x_n)$, $x_i \geq 0$, $i = 1, 2, \ . \ . \ . \ , n$, $p, q, r \geq 0$ and $p + q + r \neq 0$.

Motivated by the Bonferroni mean $B^{p,q}(X)$ and the geometric mean , Xia, Xu and Zhu [23] introduced a new mean which is called the geometric Bonferroni mean, as follows:

Key words and phrases. Schur m-power convexity, geometric Bonferroni means, majorization.

Motivated by the Bonferroni mean $B^{p,q}(X)$ and the geometric mean $G(x) = \Pi_{i=1}^{n}(x_i)^{\frac{1}{n}}$, Xia, Xu and Zhu [23] introduced a new mean which is called the geometric Bonferroni mean, as follows:

$$GB^{p,q}(X) = \frac{1}{p+q}\prod_{i,j=1,i\neq j}^{n}(px_i + qx_j)^{\frac{1}{n(n-1)}} \tag{1.3}$$

where $X = (x_1, x_2, ..., x_n)$, $x_i \geq 0$, $i = 1, 2, \ . \ . \ . \ , n$, $p, q \geq 0$ and $p + q \neq 0$. An extension of the geometric Bonferroni mean was given by Park and Kim in [19], which is called the generalized geometric Bonferroni mean, *i.e.*,

$$GB^{p,q,r}(X) = \frac{1}{p+q+r}\prod_{i,j,k=1,i\neq j\neq k}^{n}(px_i + qx_j + rx_k)^{\frac{1}{n(n-1)(n-2)}}, \tag{1.4}$$

where $X = (x_1, x_2, ..., x_n)$, $x_i \geq 0$, $i = 1, 2, \ . \ . \ . \ , n$, $p, q, r \geq 0$ and $p + q + r \neq 0$.

Remark 1.1. For $r = 0$, it is easy to observe that

$$GB^{p,q,0}(X) = \frac{1}{p+q+0}\prod_{i,j=1,i\neq j}^{n}\left[\prod_{k=1,i\neq j\neq k}^{n} px_i + qx_j + 0 \times x_k\right]^{\frac{1}{n(n-1)(n-2)}}, \tag{1.5}$$

$$= \tfrac{1}{p+q}\left[\prod_{i,j=1,i\neq j}^{n}(px_i + qx_j)^{(n-2)}\right]^{\frac{1}{n(n-1)(n-2)}},$$

$$= \tfrac{1}{p+q}\left[\prod_{i,j=1,i\neq j}^{n}(px_i + qx_j)\right]^{\frac{1}{n(n-1)}}, =GB^{p,q}(X)$$

Remark 1.2. If $q = 0$, $r = 0$, then the generalized geometric Bonferroni mean reduces to the geometric mean, *i.e.*, $GB^{p,0,0}(X) = GB^{p,0}(X) = \frac{1}{p}\prod_{i,j=1,i\neq j}^{n}(px_i)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n}(x_i)^{\frac{1}{n}} = G(X)$

Remark 1.3. If $x = (x, x, ..., x)$, then $GB^{p,q,r}(X) = GB^{p,q,r}(x, x, ..., x) = x$.

For convenience, throughout the paper R denotes the set of real numbers, $x = (x_1, x_2, ..., x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$R^n = \{X = (x_1, x_2, ..., x_n),\ x_i \varepsilon \text{R, i}= 1, 2, ..., \text{n}, \}$,

$R^n_+ = \{X = (x_1, x_2, ..., x_n),\ x_i \geq 0, \text{ i}= 1, 2, ..., \text{n}, \}$,

$R^n_{++} = \{X = (x_1, x_2, ..., x_n),\ x_i > 0, \text{ i}= 1, 2, ..., \text{n}, \}$

In a recent paper [24], Shi and Wu investigated the Schur m-power convexity of the geometric Bonferroni mean $GB^{p,q}(x)$. The definition of Schur m-power convex function is as follows:

Definition 1.1. *Let $f : R_{++} \to R$ be defined by*

$$f(x) = \begin{cases} \frac{x^m - 1}{m}, & m \neq 0 \\ \ln x, & m = 0 \end{cases} \tag{1.6}$$

Then a function $\psi : \Omega \subset R^n_{++} \to R$ is said to be Schur m- power convexity on Ω.
If $(f(x_1), (f(x_2), ...(f(x_n)) \prec (f(y_1), (f(y_2), ...(f(y_n)))$ for all $(x_1, x_2.....x_n) \in \Omega$ and $(y_1, y_2.....y_n) \in \Omega$ implies $\phi(x) \leq \phi(y)$.
If -$\phi(x)$ is Schur m power convex, then we say that $\phi(x)$ is Schur m power concave.

In recent years, the theory of majorization has been used as an important tool in studying the properties of the mean. Yang [8],[9],[10] generalized the notion of Schur convexity to Schur f-convexity and discussed the Schur m- power convexity of Stolarsky means [8], Gini means [9] and Daroczy means [10]. Subsequently, the Schur m-power convexity has evoked the interest of many researchers (see [11], [12], [13], [14]).

In this paper, we discuss the Schur m-power convexity of the generalized geometric Bonferroni mean $GB^{p,q,r}(X)$, Our main results are stated in the following theorem.

Theorem I. *Let $x = (x_1, x_2, ..., x_n) \in R^n_{++}$.For fixed $(p, q) \in R^2_{++}$ and $n \geq 3$,*

(i) if $m < 0$, then $GB^{p,q,r}(X)$ is Schur m-power convex;

(ii) if $m = 0$, then $GB^{p,q,r}(X)$is Schur m-power convex;

(iii) if $m = 1$, then $GB^{p,q,r}(X)$ is Schur m-power concave;

(iv) if $m \geq 2$, then $GB^{p,q,r}(X)$ is Schur m-power concave.

2. Preliminaries

We begin with recalling some basic concepts and notations in the theory of majorization. For more details, we refer the reader to [15, 16].

Definition 2.1. *Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in R^n$*

(1) x is majorized by y, (in symbol $x \prec y$), if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ and $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$, where $x_{[1]} \geq, ..., \geq x_{[n]}$ and $y_{[1]} \geq, ..., \geq y_{[n]}$ are rearrangements of x and y in descending order.
(2) $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2, ..., n$. Let $\Omega \in R^n (n \geq 2)$. The function $\varphi : \Omega \to R$ is said to be decreasing if and only if $-\varphi$ is increasing.
(3) $\Omega \subset R^n$ is called a convex set if $(\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n)$ for every x and $y \in \Omega$ where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.
(4) Let $\Omega \subset R^n$ the function $\varphi : \Omega \to R$ be said to be a Schur convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex.

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Definition 2.2. *(i) The set $\Omega \subset R^n$ is called symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.*

(ii) The function $\varphi : \Omega \to R$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Lemma 2.1. *Let $\Omega \subset R^n_+$ be a symmetric set with nonempty interior Ω^0 and $\varphi : \Omega \to R$ be continuous on Ω and differentiable in Ω^0. Then φ is Schur m power convex on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}[x_1^{1-m}\frac{\partial\varphi(x)}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi(x)}{\partial x_2}] \geq 0, if\ m \neq 0 \tag{2.1}$$

and

$$(lnx_1 - lnx_2)[x_1\frac{\partial\varphi(x)}{\partial x_1} - x_2\frac{\partial\varphi(x)}{\partial x_2}] \geq 0, if\ m = 0 \tag{2.2}$$

3. **Proof of Main Results**

Proof. The generalized geometric Bonferroni mean is defined by

$$GB^{p,q,r}(X) = \tfrac{1}{p+q+r}\prod_{i,j,k=1, i\neq j\neq k}^{n}(px_i + qx_j + rx_k)^{\frac{1}{n(n-1)(n-2)}},$$

taking the natural logarithm gives

$$\log GB^{p,q,r}(X) = \log\frac{1}{p+q+r} + \frac{1}{n(n-1)(n-2)}\ Q, \tag{3.1}$$

where

$$\begin{aligned} Q = & \sum_{j,k=3,j\neq k}^{n}[\log(px_1 + qx_j + rx_k) + \log(px_2 + qx_j + rx_k)] \\ & + \sum_{i,k=3,i\neq k}^{n}[\log(px_i + qx_1 + rx_1) + \log(px_i + qx_2 + rx_2)] \\ & + \sum_{i,j=3,i\neq j}^{n}[\log(px_i + qx_j + rx_1) + \log(px_i + qx_j + rx_2)] \\ & + \sum_{k=3}^{n}[\log(px_1 + qx_2 + rx_k) + \log(px_2 + qx_1 + rx_k)] \\ & + \sum_{j=3}^{n}[\log(px_1 + qx_j + rx_2) + \log(px_2 + qx_j + rx_1)] \end{aligned}$$

$$[\log(px_i + qx_1 + rx_2) + \log(px_i + qx_2 + rx_1)]$$

$$i=3$$

$$+ \sum_{i,j,k=3, j\neq k}^{n} \log(px_i + qx_j + rx_k)$$

Partial Differentiating the equation (3.1) with respect to x_1 and x_2, respectively, we have

$$\begin{aligned}
\frac{\partial GB^{p,q,r}(x)}{\partial x_1} &= \frac{GB^{p,q,r}(x)}{n(n-1)(n-2)}.\frac{\partial Q}{\partial x_1} \\
&= \frac{GB^{p,q,r}(x)}{n(n-1)(n-2)}\Bigg[\sum_{j,k=3,j\neq k}^{n}\frac{p}{px_1+qx_j+rx_k} + \sum_{i,k=3,i\neq k}^{n}\frac{q}{px_i+qx_1+rx_k} \\
&+ \sum_{i,j=3,i\neq j}^{n}\frac{r}{px_i+qx_j+rx_1} + \sum_{k=3}^{n}\left(\frac{p}{px_1+qx_2+rx_k}+\frac{q}{px_2+qx_1+rx_k}\right) \\
&+ \sum_{j=3}^{n}\left(\frac{p}{px_1+qx_j+rx_2}+\frac{r}{px_2+qx_j+rx_1}\right) \\
&+ \sum_{i=3}^{n}\left(\frac{q}{px_i+qx_1+rx_2}+\frac{r}{px_i+qx_2+rx_1}\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial GB^{p,q,r}(x)}{\partial x_2} &= \frac{GB^{p,q,r}(x)}{n(n-1)(n-2)}.\frac{\partial Q}{\partial x_2} \\
&= \frac{GB^{p,q,r}(x)}{n(n-1)(n-2)}\Bigg[\sum_{j,k=3,j\neq k}^{n}\frac{p}{px_2+qx_j+rx_k} + \sum_{i,k=3,i\neq k}^{n}\frac{q}{px_i+qx_2+rx_k} \\
&+ \sum_{i,j=3,i\neq j}^{n}\frac{r}{px_i+qx_j+rx_2} + \sum_{k=3}^{n}\left(\frac{q}{px_1+qx_2+rx_k}+\frac{p}{px_2+qx_1+rx_k}\right) \\
&+ \sum_{j=3}^{n}\left(\frac{r}{px_1+qx_j+rx_2}+\frac{p}{px_2+qx_j+rx_1}\right) \\
&+ \sum_{i=3}^{n}\left(\frac{r}{px_i+qx_1+rx_2}+\frac{q}{px_i+qx_2+rx_1}\right)\Bigg],
\end{aligned}$$

It is easy to see that $GB^{p,q,r}(x)$ is symmetric on R^n_+. Without loss of generality, we may assume that $x_1 \geq x_2$.

Direct computation gives

$$\begin{aligned}
\triangle &= \frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial\ GB^{p,q,r}(x)}{\partial x_1} - x_2^{1-m}\frac{\partial\ GB^{p,q,r}(x)}{\partial x_2}\right) \\
&= \frac{(x_1^m - x_2^m)GB^{p,q,r}(x)}{mn(n-1)(n-2)}\Bigg[p\sum_{j,k=3,j\neq k}^{n}\left(\frac{x_1^{1-m}}{px_1+qx_j+rx_k} - \frac{x_2^{1-m}}{px_2+qx_j+rx_k}\right)
\end{aligned}$$

$$+ q \sum_{r,k=3,r\neq k}^{n} \left(\frac{x_1^{1-m}}{px_i + qx_1 + rx_k} - \frac{x_2^{1-m}}{px_i + qx_2 + rx_k} \right)$$
$$+ r \sum_{i,j=3,i\neq j}^{n} \left(\frac{x_1^{1-m}}{px_i + qx_j + rx_1} - \frac{x_2^{1-m}}{px_i + qx_j + rx_2} \right)$$
$$+ \sum_{k=3}^{n} \left(\frac{px_1^{1-m} - qx_2^{1-m}}{px_1 + qx_2 + rx_k} + \frac{qx_1^{1-m} - px_2^{1-m}}{px_2 + qx_1 + rx_k} \right)$$
$$+ \sum_{j=3}^{n} \left(\frac{px_1^{1-m} - rx_2^{1-m}}{px_1 + qx_j + rx_2} + \frac{rx_1^{1-m} - px_2^{1-m}}{px_2 + qx_j + rx_1} \right)$$
$$+ \sum_{i=3}^{n} \left(\frac{qx_1^{1-m} - rx_2^{1-m}}{px_i + qx_1 + rx_2} + \frac{rx_1^{1-m} - qx_2^{1-m}}{px_i + qx_2 + rx_1} \right) \Bigg]$$

$$\triangle = \frac{(x_1^m - x_2^m) GB^{p,q,r}(x)}{mn(n-1)(n-2)} \Bigg[p \sum_{j,k=3,j\neq k}^{n} \left(\frac{(x_1^{-m} - x_2^{-m}) px_1x_2 + (qx_j + rx_k)(x_1^{1-m} - x_2^{1-m})}{(px_1 + qx_j + rx_k)(px_2 + qx_j + rx_k)} \right)$$
$$+ q \sum_{r,k=3,r\neq k}^{n} \left(\frac{(x_1^{-m} - x_2^{-m}) qx_1x_2 + (px_i + rx_k)(x_1^{1-m} - x_2^{1-m})}{(px_i + qx_1 + rx_k)(px_i + qx_2 + rx_k)} \right)$$
$$+ r \sum_{r,k=3,r\neq k}^{n} \left(\frac{(x_1^{-m} - x_2^{-m}) rx_1x_2 + (px_i + qx_j)(x_1^{1-m} - x_2^{1-m})}{(px_i + qx_j + rx_1)(px_i + qx_j + rx_2)} \right)$$
$$+ \sum_{k=3}^{n} \left(\frac{2pq(x_1^{2-m} - x_2^{2-m}) + (rx_k)(p+q)(x_1^{1-m} - x_2^{1-m}) + (p^2+q^2)x_1x_2(x_1^{-m} - x_2^{-m})}{(px_1 + qx_2 + rx_k)(px_2 + qx_1 + rx_k)} \right)$$
$$+ \sum_{j=3}^{n} \left(\frac{2pr(x_1^{2-m} - x_2^{2-m}) + (qx_j)(p+r)(x_1^{1-m} - x_2^{1-m}) + (p^2+r^2)x_1x_2(x_1^{-m} - x_2^{-m})}{(px_1 + qx_j + rx_2)(px_2 + qx_j + rx_1)} \right)$$
$$+ \sum_{i=3}^{n} \left(\frac{2qr(x_1^{2-m} - x_2^{2-m}) + (px_i)(q+r)(x_1^{1-m} - x_2^{1-m}) + (q^2+r^2)x_1x_2(x_1^{-m} - x_2^{-m})}{(px_i + qx_1 + rx_2)(px_i + qx_2 + rx_1)} \right) \Bigg]$$

If $m < 0$, then $x_1^m - x_2^m \leq 0$, $x_1^{-m} - x_2^{-m} \geq 0$, $x_1^{1-m} - x_2^{1-m} \geq 0$ and $x_1^{2-m} - x_2^{2-m} \geq 0$. Thus $\triangle \geq 0$. From Lemma 2.1, it follows that $GB^{p,q,r}(x)$ is Schur m - power convex for $x \in R_{++}^n$.

If $m \geq 2$, then $x_1^m - x_2^m \geq 0$, $x_1^{-m} - x_2^{-m} \leq 0$, $x_1^{1-m} - x_2^{1-m} \leq 0$ and $x_1^{2-m} - x_2^{2-m} \leq 0$. Thus $\triangle \leq 0$ From Lemma 2.1, it follows that $GB^{p,q,r}(x)$ is Schur m - power concave for $x \in R_{++}^n$.

If $m=1$, Then

$$\triangle = \frac{-(x_1-x_2)^2 GB^{p,q,r}(x)}{n(n-1)(n-2)} \Bigg[\sum_{j,k=3,j\neq k}^{n} \left(\frac{p^2}{(px_1+qx_j+rx_k)(px_2+qx_j+rx_k)} \right) + \sum_{r,k=3,r\neq k}^{n} \left(\frac{q^2}{(px_i+qx_1+rx_k)(px_i+qx_2+rx_k)} \right) + \sum_{r,k=3,r\neq k}^{n} \left(\frac{r^2}{(px_i+qx_j+rx_1)(px_i+qx_j+rx_2)} \right) + \sum_{k=3}^{n} \left(\frac{(p-q)^2}{(px_1+qx_2+rx_k)(px_2+qx_1+rx_k)} \right) + \sum_{j=3}^{n} \left(\frac{(p-r)^2}{(px_1+qx_j+rx_2)(px_2+qx_j+rx_1)} \right) + \sum_{i=3}^{n} \left(\frac{(q-r)^2}{(px_i+qx_1+rx_2)(px_i+qx_2+rx_1)} \right) \Bigg] \leq 0$$

. By using Lemma 2.1 $GB^{p,q,r}(x)$ is Schur m - power concave for $x \in R^n_{++}$.

If $m=0$, then

$$\triangle = \frac{(x_1-x_2)(\log x_1 - \log x_2) GB^{p,q,r}(x)}{n(n-1)(n-2)} \Bigg[p \sum_{j,k=3,j\neq k}^{n} \left(\frac{qx_j+rx_k}{(px_1+qx_j+rx_k)(px_2+qx_j+rx_k)} \right) + q \sum_{r,k=3,r\neq k}^{n} \left(\frac{px_i+rx_k}{(px_i+qx_1+rx_k)(px_i+qx_2+rx_k)} \right) + r \sum_{r,k=3,r\neq k}^{n} \left(\frac{px_i+qx_j}{(px_i+qx_j+rx_1)(px_i+qx_j+rx_2)} \right) + \sum_{k=3}^{n} \left(\frac{2(x_1+x_2)pq + rx_k(p+q)}{(px_1+qx_2+rx_k)(px_2+qx_1+rx_k)} \right) + \sum_{j=3}^{n} \left(\frac{2(x_1+x_2)pr + qx_j(p+r)}{(px_1+qx_j+rx_2)(px_2+qx_j+rx_1)} \right) + \sum_{i=3}^{n} \left(\frac{2(x_1+x_2)qr + px_i(q+r)}{(px_i+qx_1+rx_2)(px_i+qx_2+rx_1)} \right) \Bigg] \geq 0$$

. By using Lemma 2.1 we conclude that $GB^{p,q,r}(x)$ is Schur m - power convex for $x \in R^n_{++}$.

The proof of Theorem I is completed.

□

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