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Distance-2 Domination Integrity of Zero Divisor Graph over Ring Z_n

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Abstract:A non-empty subset D of vertices in a graph G = (V, E) is a distance - 2 dominating integrity set if every vertex in V-D is within distance - 2 from at least one vertex in D and is defined as $D_{\leq 2}$ I(G) = min{ | X | +m(G-X) : X is a distance-2 dominating set where m(G-X) is the order of a maximum connected component of G-X. The hub of this article is a search on the behavior of the distance-2 domination integrity of zero divisor graphs and cartesian product of two star zero divisor graphs.

Key Words: Integrity, Domination integrity, Distance -2 domination integrity, Zero divisor graph.

AMS Subject Classification: 05C76, 05C69.

1. INTRODUCTION:

A simple graph is a pair G = (V,E), where V is the vertex set and E is the edge set. In fact, V can be any finite set and E consists of unordered pairs {u, v} of distinct elements of V . When two vertices are connected by an edge they are said to be adjacent.[3] Some graphs can be constructed with special elements of a ring R, where R is a commutative ring and with an identity element. Let R be a Commutative ring with unit element and let Z(R) be its set of zero divisors. The zero divisor graph of R is denoted by $[\Gamma(Z(R))]$ and it is defined as follows: A two distinct vertices x and y are adjacent iff xy =0 for all x,y in Z(R)\{0}. All the graphs considered here are simple, finite, connected and undirected graph. The concept of the zero divisor graph was first introduced by I.Beck in 1988 and further developed by D.D.Anderson and M.Naseer.

The vulnerability of network has been studied in various contexts including road transportation system, information security, structural engineering and communication network.

A graph structure is vulnerable if ‘any small damage produces large consequences’. In a communication network, the vulnerability measures the resistance of the network to disruption of operation s after the failure of certain stations (junctions) or communication links (connections). In the theory of graphs, the vulnerability implies a lack of resistance (weakness) of graph network arising from deletion of vertices or edges or both. Communication networks must be so designed that they do not easily get disrupted under external attack and even if they get disturbed then they should be easily reconstructible.

2. BASIC DEFINITIONS AND PRELIMINARIES

In this section, we discussed some basic definitions ,notations and their meanings

Definition 2.1[6]

A field is a commutative ring where every non-zero element 'a' is invertible. That is, if R has a multiplicative inverse 'b' such that ab = 1 then by definition, any field is a commutative ring.

Definition 2.2 [7]

A zero divisor of a commutative ring R is a non-zero element 'r' such that, rs = 1for some other non-zero element s of the ring R. If the ring R is commutative, then rs = 0 $\Leftrightarrow$ sr = 0.

Definition 2.3 [6]

(Beck) The zero divisor graph of a commutative ring R with unit element is a simple graph whose set of vertices consists of all elements of the ring, with an edge defined between 'a' and 'b' if and only if ab = 0.

Definition 2.4 [2]

In graph theory, the Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 is a graph such that,

- The vertex set of $G_1 \times G_2$ is the Cartesian product V(G_1) × V(G_2) and
- Any two vertices (u ,u^1) and (v,v^1) are adjacent in $G_1 \times G_2$ if and only if either,
  i. u = v and u^1 is adjacent with v^1 in G_2.
  ii. u^1= v^1 and u adjacent with v in G_1.

Definition 2.5[10]

The integrity of a graph G is denoted by $I(G)$ and defined by $I(G) = \min\{|S| + m(G-S) : S \subset V(G)$, where $m(G-S)$ is the order of a maximum component of $G-S$.

Definition 2.6[10]

A non empty set $D \subseteq V(G)$ is said to be a dominating set of G if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The domination number $\gamma(G)$ is the minimum cardinality taken over all the minimal dominating sets of G.

Definition 2.7[10]

The domination integrity of a connected graph G denoted by $DI(G)$ and defined as $DI(G) = \min\{|X| + m(G-X) : X$ is a dominating set }where $m(G - X)$ is the order of a maximum component of $G - X$.

3. MAIN RESULTS

In this section some results on distance-2 domination integrity of zero divisor graphs are presented.

Theorem 3.1 [3]

If n = 2p where p is an odd prime, then the non-zero zero divisor graph is a star graph and it is also called complete bipartite graph ($K_{1,p-1}$).

Theorem 3.2 [3]

If n = 2p where p is an odd prime, then the dominating number of star zero divisor graph is 1.

Observation 3.3

Every zero divisor graph of $\Gamma(Z_n)$ is a distance-2 dominating set.

Theorem 3.4

For $\Gamma(Z_n)$, if $n = 2p$ where p is an odd prime number then $D_{\le 2}I\,(\Gamma(Z_n)) = 2$

Proof

If $n = 2p$. By above theorem 3.2 D is a dominating set of $\Gamma(Z_n)$ and $V - D \in N(D)$. Since every vertex in $V - D$ has a distance at the most 2. Let D be the distance -2 dominating set of $\Gamma(Z_n)$ then every vertex in V-D has a distance one. Obviously $|D_{\le 2}|$=1 and $m(G - D) = 1$. Here $|D_{\le 2}| = m(G - D)$

Therefore, $\min\{|X| + m(G - X) : X$ is a distance-2 dominating set $= |D_{\le 2}| + m(G - D)$

=1+1

=2

Hence, $D_{\le 2}I\,(\Gamma(Z_n)) = 2$

Theorem 3.5

For $\Gamma(Z_n)$, if $n = 3p$ where p is an prime number then $D_{\le 2}I\,(\Gamma(Z_n)) = \begin{cases} 2 & if\ \ p = 2,3 \\ p+1 & if\ p > 3 \end{cases}$

Proof

Case(i) If p=2,3 then by the above theorem 3.4

$$D_{\le 2}I\,(\Gamma(Z_n)) = 2$$

Case(ii)

Subcase(i) If p>3, then n=15 and we have

$$G = \Gamma(Z_{15})$$

$V(G) = \{3,5,6,9,10,12\}$ and its order is equal to 6.

The set of pair of adjacent vertices in V(G)={(3,10),(6,10),(12,10),(9,10),(3,5),(6, 5),(12,5),(9,5}

The set of pair of adjacent vertices in V(G)={(3,6),(6,12),(12,9),(3,12),(3,9),(6,9),(12,9),(10,5)} The dominating set of G={(3,10),(3,5)}forms a minimal dominating set with cardinality 2.

By the above observation 3.3

$$D_{\le 2}I\,(\Gamma(Z_n)) = 1$$

and m(G-D)=5

Now we discuss the minimality of |D|+m(G-D). If there does not exist any distance-2 dominating set of $D_1$of G such that $|D_1|$=|D| and m(G- D_1)=5. It can be checked that for any distance-2 dominating set and m(G-D_2)=5.

Therefore, $\min\{|X| + m(G - X) : X$ is a distance-2 dominating set $= |D_{\le 2}| + m(G - D)$

=1+5

=6

Hence, $D_{\le 2}I\,(\Gamma(Z_n)) = 6$

Sub case (ii)

If p=3, then n=21and we have

$$G = \Gamma(Z_{21})$$

$V(G) = \{3,6,7,9,12,15,14,18\}$ and its order equal to 8.

The set of pair of adjacent vertices in V(G)={(7,3),(7,6),(7,9),(7,12),(7,15),

(7,18),
(14,3),(14,6),(14,9),(14,12),(14,15),(14,18)}

The set of pair of non-adjacent vertices in V(G)={(3,6),(3,9),(3,12),(3,15),(3,18),(6,9),

(6,12),(6,15),(6,18),(9,12),(9,15),(9,18),(12, 15),

(14,18),(15,18),(7,4)}

The dominating set of G is ={(7,3),(14,12} forms a minimal dominating set.

By the above observation 3.3 |D|= 1 forms a distance-2 minimal dominating set with m(G-D)=7

Therefore, $\min\{|X| + m(G - X) : X$ is a distance-2 dominating set $= |D_{\leq 2}| + m(G - D)$

=1+7

=6

Hence, $D_{\leq 2}I\left(\Gamma(Z_n)\right) = p+1$

Theorem 3.6

For $\Gamma(Z_n)$, if $n = pq$, where p and q odd prime numbers and $p < q$, then

$$D_{\leq 2}I(\Gamma(Z_n)) = \begin{cases} 6 & if\ p = 3, q = 5 \\ 10 & if\ p = 5, q = 7 \\ 15 & if\ p = 7, q = 11 \end{cases}$$

Proof

Case(i)

If p=3, q=5 then we have the graph given by

$$G = \Gamma(Z_{3(5)}) = \Gamma(Z_{15})$$

By the above theorem 3.5 subcase(i) it is factual.

Case(ii)

If p=5, q=7 we have

$$G = \Gamma(Z_{3(5)}) = \Gamma(Z_{15})$$

Since,

$V(G) = \{5,7,10,14,15,20,21,25,28,30\}$

$|V(G)| = 10$

The set of pair of adjacent vertices={(7,5),(7,10),(7,15),(7,20),(7,25),(7,30),

(14,5),(14,10),(14,15),(14,20),(14,25),(14,30),

(21,5),(21,10),(21,15),(21,20),(21,25),(21,30),

(28,5),(28,10),(28,15),(28,20),(28,25),(28,30)}

The set of pair of non- adjacent vertices={(7,14),(7,21),(7,28),(5,10),(5,15),(5,20),(5,25),(5,30),

(10,15),(10,20),(10,25),(10,30),(15,20),(15,25),(15,30)

(20,25),(20,30),(15,30)

The dominating set of G is D={7,15} forms a minimal dominating set with cardinality 2.

By the above observation 3.3 |D|= 1 forms a distance-2 minimal dominating set with m(G-D)=9. Now we discuss the minimality of |D|+m(G-D). If there does not exist any distance-2 dominating set of D_1 of G such that |D|=D and m(G- D_1)=9.

Therefore, $\min\{|X| + m(G - X) : X$ is a distance-2 dominating set $= |D_{\leq 2}| + m(G - D)$

=1+9

=10

Similar results for $p = 7, q = 11$ we get $D_{\leq 2}I(\Gamma(Z_n)) = 15$

Theorem 3.7

For $\Gamma(Z_n)$, if $n = p^4$ where p is an even prime number then $D_{\leq 2}I\left(\Gamma(Z_n)\right) = 3$

Proof

If $n = p^4$ where p is an even prime then we have

$$G = \Gamma(Z_{2^4}) = \Gamma(Z_{16})$$

Since $V(G) = \{2,4,6,8,10,12,14\}$

$|V(G)| = 7$

By the above observation 3.3 |D|= 1 forms a distance-2 minimal dominating set with m(G-D)=2.

Therefore, $\min\{|X| + m(G - X) : X$ is a distance-2 dominating set $= |D_{\leq 2}| + m(G - D)$

=1+2

=3.

Theorem 3.8

Let p and q are non-distinct $(n < 100)$ primes then $D_{\leq 2}I\,(\Gamma(Z_n) = 2m \; if \; m = 1,2,3$

Theorem 3.9

For $\Gamma(Z_n)$, if $n = p^3$ where p is an even prime numbers then $D_{\leq 2}I\,(\Gamma(Z_n) = 2$

Theorem 3.10

For $\Gamma(Z_n)$, if $n = p^2q^2$ where p and q are prime numbers$(p^2q^2 < 100)$ then $D_{\leq 2}I\,(\Gamma(Z_n) = 17$

Theorem 3.11

For $\Gamma(Z_n)$, if $n = pqr(n < 50$ where p,q and r are prime number then $D_{\leq 2}I$ $(\Gamma(Z_n) = \begin{cases} 13 & if \; n = 30 \\ 17 & if \; n = 42 \end{cases}$

4. DISTANCE-2 DOMINATION INTEGRITY OF CARTESIAN PRODUCT OF THE STAR ZERO DIVISOR GRAPHS

Theorem 4.1

If $G_1 = \Gamma(Z_{2p_1})$ and $G_2 = \Gamma(Z_{2p_2})$ are two star zero divisor graphs. Let $G = G_1 \times G_2$ be the Cartesian product of star zero divisor graphs. Then $DI(G) = 2\gamma(G)$ if $p_1 < p_2$ when p_1 and p_2 are prime numbers.

Proof

Let $G_1 = \Gamma(Z_{2p_1})$ and $G_2 = \Gamma(Z_{2p_2})$ be two star zero divisor graphs.

And $\gamma(G_1) = \gamma(G_2) = 1$ by theorem 3.2,

Let $G = G_1 \times G_2$, when $p_1 < p_2$ and p_1 and p_2 are prime numbers.

Let $p_1 = 2$ and $p_2 = 3$.

We have,

$$G = \Gamma(Z_{2(2)} \times Z_{2(3)}) = \Gamma(Z_4 \times Z_6) = \Gamma(Z_4) \times \Gamma(Z_6)$$

Since, $V(G_1)$ = { 2 }. The dominating set $D_1 \subseteq V(G)$ and $D_1 = \{2\}$

Also, $|\,V(G_1)| = 1$.

$V(G_2)$ = { 2,3,4 }. The dominating set D_2 of G_2 is $D_2 = \{3\}$

And $|\,V(G_2)| = 3$.

Now, the vertex set of G is given by,

$V(G) = \{\,(2,2)\,,(2,3)\,,(2,4)\,\}$ and its order is 3.

By the definition 2.3,

The set of pair of adjacent vertices in $V_1(G) = \{\,((2,2)\ (2,3)),((2,3)\ (2,4))\,\}$

The set of non-adjacent vertices in $V_2(G) = \{\,(2,2)\,,(2,4)\,\}$. (by definition of Non-zero Zero divisor graph)

The dominating set of G is $D = \{\,(2,3)\,\}$ forms a minimal dominating set with cardinality 1(i.e) $|D| = 1$ and $m(G - D) = 1$.

Successively, $|D| = p_1$ and $m(G - D) = p_1$.

Now we discuss the minimality of $|D|$ $+m(G - D)$ there does not exist any dominating set of D_1 of G such that $|D_1| < |D|$ and $m(G - D_1) = p_1$.

Therefore, $\min\{|X| + m(G - X) : X$ is a dominating set $=|D|+ m(G - D)$

$= 2p_1$ (Here $p_1 = \gamma(G)$

Hence, $DI(G) = 2\gamma(G)$

Theorem 4.2

If $G_1 = \Gamma(Z_{2p_1})$ and $G_2 = \Gamma(Z_{2p_2})$ are two star zero divisor graphs. Let $G = G_1 \times G_2$ be the Cartesian product of star zero divisor graphs. Then

$D_{\le 2}I(G) = \begin{cases} 2 & if\ p_1 = 2, p_2 = 3 \\ p_2\gamma(G) & if\ \ p_1 < p_2 \end{cases}$ where p_1 and p_2 are prime numbers.

Proof

Case (i)

$If\ p_1 = 2, p_2 = 3$ by theorem 5.2 we have $DI(G) = 2$. Since every vertex in $V - D$ has a distance at the most 2. Obviously $|D_{\le 2}|$=1 and $m(G - D) = 1$. Here $|D_{\le 2}|= m(G - D)$

Therefore, $\min\{|X| + m(G - X) : X$ is a distance-2 dominating set $=|D_{\le 2}|+ m(G - D)$

=1+1

=2

Hence, $D_{\le 2}I(G) = 2$

Case (ii)

If $p_1 < p_2$. To find, the value of $\gamma(G_1 \times G_2)$. This can be done by using induction hypothesis,

For $p_1 = 3$ and $p_2 = 5$

We have,

$$G = \Gamma(Z_{2(3)} \times Z_{2(5)}) = \Gamma(Z_6 \times Z_{10}) = \Gamma(Z_6) \times \Gamma(Z_{10})$$

Since, | V(G_1)| $= 3 = p_1$ and D(G_1) = { 3 }

| V(G_2)| $= 5 = p_2$ and D(G_2) = { 5 }

The vertex set of G is given by

$$V(G) = \begin{Bmatrix} (2,2), (2,4), (2,5), (2,6), (2,8) \\ (3,2), (3,4), (3,5), (3,6), (3,8) \\ (4,2), (4,4), (4,5), (4,6), (4,8) \end{Bmatrix}$$

and its order is 15.

Any two vertices (u, v) $(ie., u = v = \{ 2,3,4 \} \in \Gamma(Z_6))$ and (u^1, v^1) $(\text{ie.}, u^1 = v^1 = \{ 2,4,6,5,8 \} \in \Gamma(Z_{10}))$ are adjacent in $\Gamma(Z_6 \times Z_{10})$ if and only if,

i. u = v in Z_6 and every odd integer u^1 adjacent with even integer v^1 in Z_{10}.
ii. $u^1 = v^1$ in Z_{10} and every odd integer u adjacent with even integer v in Z_6.

The set of pair of adjacent vertices in G is given by,

$$V_1(G) = \begin{cases} ((2,2)\ (2,5)) \ldots \ldots ((2,4)\ (2,5)) \ldots \ldots ((2,8)\ (2,5)) \ldots \\ ((3,2)\ (3,5)) \ldots \ldots ((3,4)\ (3,5)) \ldots \ldots ((3,8)\ (3,5)) \ldots \\ ((4,2)\ (4,5)) \ldots \ldots ((4,8)\ (4,5)) \ldots \ldots ((2,2)(3,2)) \ldots \\ ((2,8)\ (3,8)) \ldots \ldots \ldots \ldots ((4,2)(3,2)) \ldots \ldots ((4,8)(3,8)) \end{cases}$$

And $|\, v_1(G)\,| = 22$

The set of pair of non-adjacent vertices in G is given by,

$V_2(G) = V(G) - V_1(G)$

Then D = { (2,3) (3,5) (4,5) } forms a minimal dominating set with cardinality 3.

Hence,

$$\gamma(G) = \gamma(\Gamma(Z_6 \times Z_{10})) = \gamma(\Gamma(Z_6) \times \Gamma(Z_{10}))$$

$$\gamma(G) = 3 = p_1$$

The Distance-2 dominating set of G is
$D = \{ (3,5) \}$

And

$$V - D = \left\{ \begin{matrix} (2,2)\ (2,4)\ (2,6)\ (2,8), (2,5) \\ (3,2)\ (3,4)\ (3,6)\ (3,8) \\ (4,2)\ (4,4), (4,5), (4,6)\ (4,8) \end{matrix} \right\}$$

Since every vertex in $V - D$ has a distance at the most 2. Obviously $|D_{\leq 2}|$=1 and $m(G - D) = 14$.

Therefore, $\min\{|X| + m(G - X) : X$ is a distance-2 dominating set $=|D_{\leq 2}| + m(G - D)$

=1+14

=15

=5x3

Hence, $D_{\leq 2} I(G) = p_1 \gamma(G)$

5. APPLICATION OF DISTANCE-2 DOMINATION OF A ZERO DIVISOR GRAPH IN WI-FI SIGNALS

This Section deals with a real life application of distance-2 domination of a zero divisor graph in Wireless Fidelity signals.

5.1 Distance-2 Domination of a Zero Divisor Graph in Wi-Fi signals

An Educational Institution has 5 blocks namely Gandhiji block, Nehru block, Kaveri block, Kamarajar block, and Abdulkalam block. An Abdulkalam block covers all maintenance, office work, Controller of Examination, and etc. Each block contains many classes, departments, and laboratories. Administrative block has main Wi-Fi connection in its own. In this administrative block, the main Wi-Fi connection server is available which serves the Wi-Fi signal connection to each blocks simultaneously but the range varies from strong to weak for the blocks because of the distance between main server and blocks is viable in this real situation.

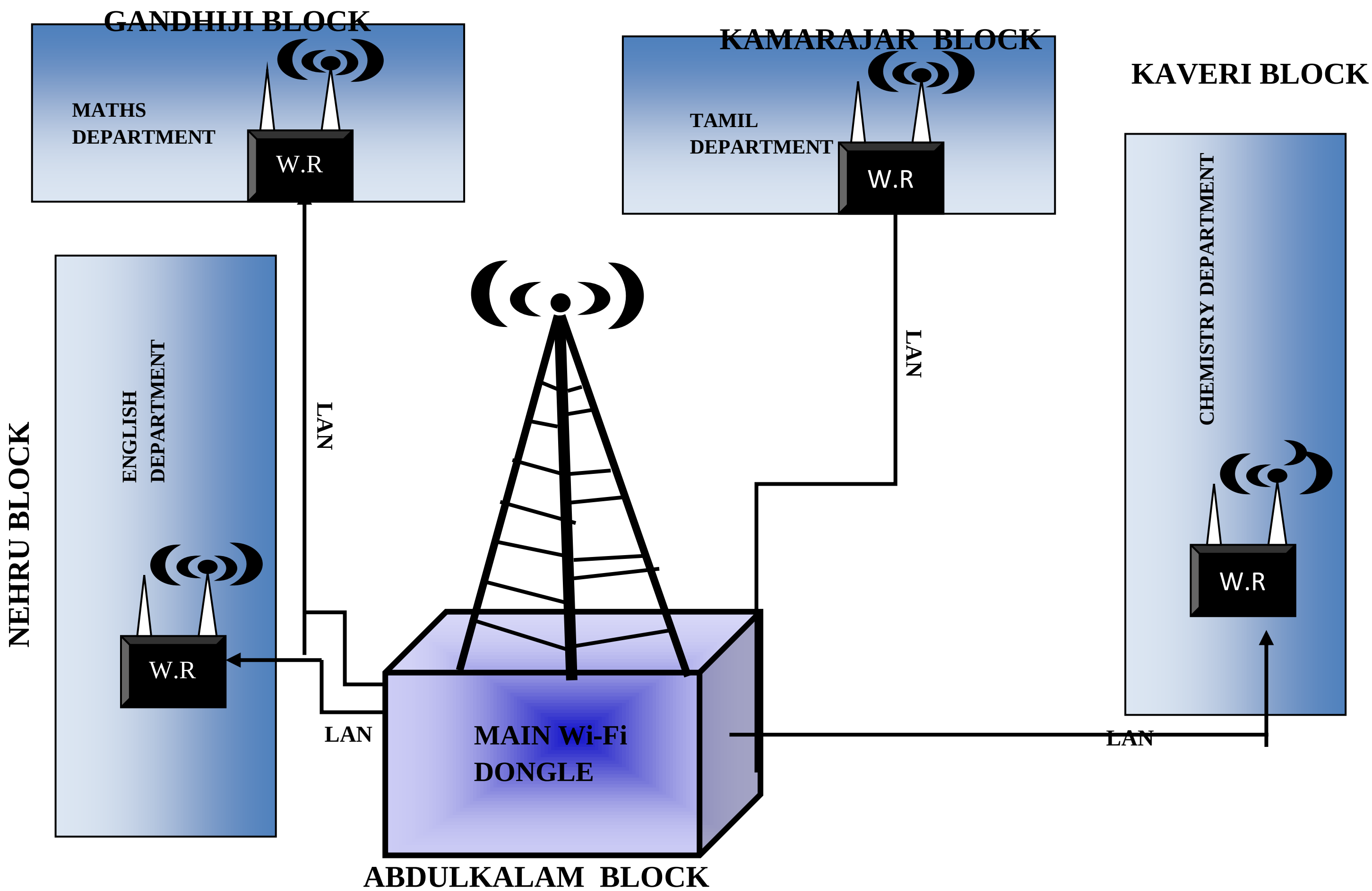
We have given a better solution for this fluctuating signal connections running in the college blocks for getting good range of full strength connection of Wi-Fi to each block in the college campus. Through this project, we described how to give the good signal strength to all of the blocks near-by using main Wi-Fi Dongle and Local Area Network (LAN) cables. We can supply good Wi-Fi connection to each block through LAN cables and Wi-Fi Router (W.R).

In Gandhiji block, we can choose department of ‘Mathematics’. In Nehru block, we can choose ‘English’ department. In Kaveri block, we can choose ‘Chemistry’ department. In Kamarajar block, we can choose ‘Tamil’ department, respectively.

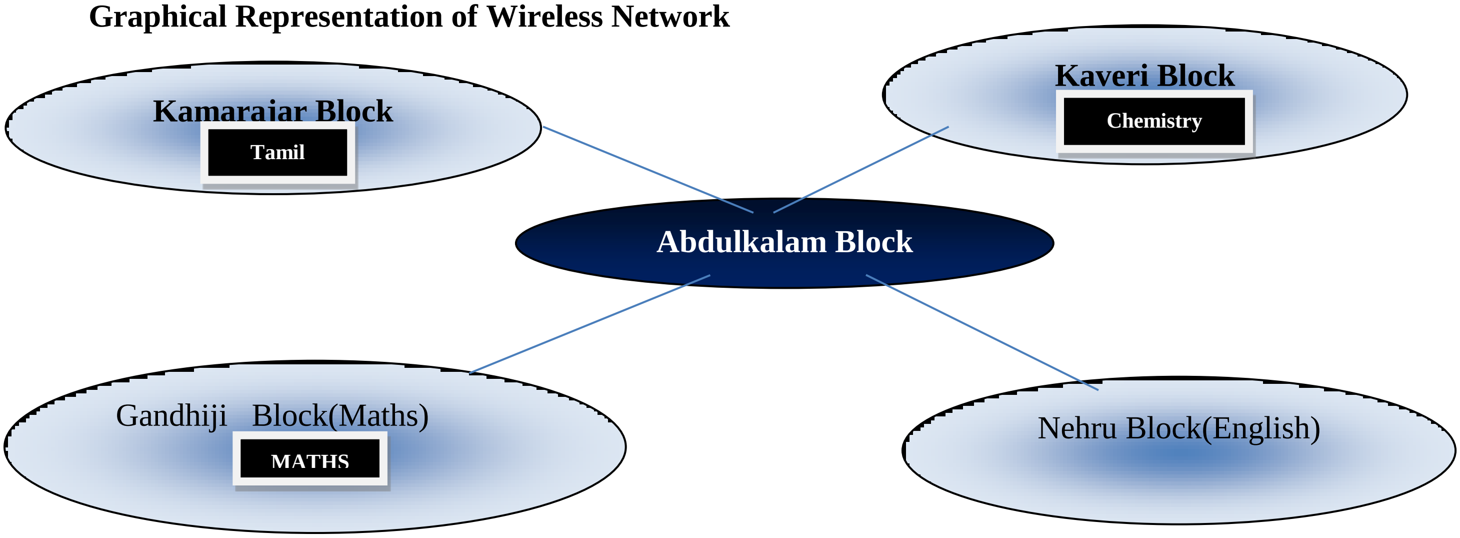
All of the 4 blocks have their own chosen departments. In each department, we fix the sub Wi-Fi Router for better signal transportation. So, the signal will pass from main server to the sub Wi-Fi Router in each block. From that sub Wi-Fi Router, other departments in that block will have their connections properly, so that every students and staffs of the Institution will get their proper good Wi-Fi signal connection for their study purpose. The purpose of the sub Wi-Fi Router connection is to provide good range of Wi-Fi connection to all the classes and departments in blocks.

Based on star zero divisor graph $\Gamma(Z_{10})$, Wi-Fi Router gives proper signal to their adjacent blocks. The graphical representation of a Wireless network described the star zero divisor graph. It has 5 vertices, each vertex represents one block. Since the Abdulkalam block is adjacent to every other blocks namely Gandhiji block, Nehru block, Kaveri block, and Kamarajar block, Wi-Fi router produces the proper Wi-Fi signal to the specified department which is of distance at the most 2. Also Abdulkalam block dominates all the 4 blocks and its distance-2 domination number is one.

Wireless Network in an Educational Institution



Graphical Representation of Wireless Network



The rapid growth of various modes of communication have emerged as a search for sustainable and secured network. The vulnerability of network is an important issue with special reference to defence objectives. We take up this problem in the context of expansion of graph network by means of various Zero divisor of line graphs, central graphs, Total graphs and Middle graphs and investigate domination integrity of resultant graphs.

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