International Journal of Research in Advent Technology, Vol.7, No.1, January 2019
E-ISSN: 2321-9637
Available online at www.ijrat.org

Analysis of complex roots of the equation $x^3+(k-1)x^2-(k-1)x+k=0$ through complex continued fraction, where $k\geq 1$

Krithika S.[1] and Gnanam A.[2]
Assistant professor[1], Assistant professor[2]
[1]Department of Mathematics, Seethalakshmi Ramaswami College, Trichy-2.
[2]Department of Mathematics, Government Arts College, Trichy-22.
Email:krithikasrc@gmail.com[1],gnaanam@yahoo.com[2]

Abstract

Complex roots of a polynomial always occur in pairs. To solve a cubic equation using Newton-Raphson method is more advantageous than any other method. Here in this paper an attempt has been made to find the complex roots of the equation $x^3+(k-1)x^2-(k-1)x+k=0$ through complex continued fraction also a comparison is made regarding number of iterations when $k=1$.

Keywords: Continued fraction, Finite complex continued fraction, Infinite complex continued fraction, Periodic complex continued fraction.

Subject Classification: MSC 11A05, 11A55, 11J70, 11Y65, 30B70, 40A15.

1.INTRODUCTION

The Indian mathematician Aryabhata used a continued fraction to solve a linear indeterminate equation. For more than a thousand years, any work that used continued fractions was restricted to specific examples. Throughout Greek and Arab mathematical writing, we can find examples and traces of continued fractions. Euler showed that every rational can be expressed as a terminating simple continued fraction. He also provided an expression for e in continued fraction form. He used this expression to show that e and e^2 are irrational [4].

Continued fraction plays an important role in number theory. It is used to represent the rational numbers to an another form by using Euclidean algorithm.

An expression of the form

$$\frac{p}{q}=a_0+\cfrac{b_0}{a_1+\cfrac{b_1}{a_2+\cfrac{b_2}{a_3+\cfrac{b_3}{\ddots}}}}$$

where a_i, b_i are real or complex numbers is called a continued fraction [6].

An expression of the form

$$a_0+\cfrac{e_0}{a_1+\cfrac{e_1}{a_2+\cfrac{e_2}{a_3+\cfrac{e_3}{\ddots+\cfrac{e_n}{a_n+\ddots}}}}}$$

where $a_0, a_1, a_2, a_3, \cdots$ are in $Z(i)$ and $e_i's$ are units of complex numbers, $e_k \in \{1,-1,i,-i\}, k=1,2,3,...$ is known as a complex continued fraction [5].

The complex continued fraction is commonly expressed as $a_0+\frac{e_1}{a_1+}\frac{e_2}{a_2+}\frac{e_3}{a_3+}\cdots$.

The quantities $a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots$ are called the elements of the complex continued fraction.

In a finite complex continued fraction the number of elements are finite, where as an infinite continued fraction have infinite number of quantities.

Therefore $a_0+\frac{e_1}{a_1+}\frac{e_2}{a_2+}\frac{e_3}{a_3+}\cdots\frac{e_n}{a_n}$ is known as finite complex continued fraction and an expression $a_0+\frac{e_1}{a_1+}\frac{e_2}{a_2+}\frac{e_3}{a_3+}\cdots\frac{e_n}{a_n+}\frac{e_{n+1}}{a_{n+1}+}\cdots$ is known as an infinite complex continued fraction.

In a finite or infinite complex continued fraction $a_0, a_1, a_2, a_3, \cdots$ are called the partial quotients and $e_1, e_2, e_3, \cdots$ are known as the partial numerators.

The length of a finite complex continued fraction $a_0+\frac{e_1}{a_1+}\frac{e_2}{a_2+}\frac{e_3}{a_3+}\cdots\frac{e_n}{a_n}$ is $n+1$ and the length of a infinite continued fraction $a_0+\frac{e_1}{a_1+}\frac{e_2}{a_2+}\frac{e_3}{a_3+}\cdots\frac{e_n}{a_n+}\frac{e_{n+1}}{a_{n+1}+}\cdots$ is ∞.

The value of the finite complex continued fraction is denoted as $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_n}{a_n}\right]$.

2. PROPERTIES AND ALGORITHM OF COMPLEX CONTINUED FRACTION:

2.1 Convergence of complex continued fraction:[1]

The successive convergence of the complex continued fractions are $[a_0]$, $\left[a_0, \frac{e_1}{a_1}\right]$, $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}\right]$ and so on and they are denoted by $c_0=\frac{p_0}{q_0}$, $c_1=\frac{p_1}{q_1}$, $c_2=\frac{p_2}{q_2}$,… In general the *nth* convergent is denoted by $c_n=\frac{p_n}{q_n}=\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_n}{a_n}\right]$, where $p_i's$ and $q_i's$ are called the numerators and denominators of convergent of the complex continued fraction.

2.2 Properties of complex continued fraction:[1]

Let $\left[a_0, \frac{e_1}{a_1}, \frac{e_2}{a_2}, \frac{e_3}{a_3}, \cdots \frac{e_n}{a_n}, \cdots\right]$ be an infinite complex continued fraction. We inductively define two infinite sequences p_k and q_k, $k \geq 1$ by

$$p_0 = a_0 \qquad p_{-1}=1$$
$$q_0 = 1 \qquad q_{-1}=0$$

$$p_k = a_k\, p_{k-1} + e_k\, p_{k-2}, k \geq 1$$
$$q_k = a_k\, q_{k-1} + e_k\, q_{k-2}, k \geq 1$$

2.3 Algorithm of complex continued fraction:[2,5,7]

Let $x \in C$. Suppose we wish to find continued fraction expansion of x. Take $x_0 = \text{Re}(x) + i\,\text{Im}(x)$.

Let $a = \text{Re}(x) - [\text{Re}(x)]$ and $b = \text{Im}(x) - [\text{Im}(x)]$ so that $a < 0$ and $b < 0$.

Now $[x_0] = [\text{Re}(x)] + i[\text{Im}(x)] + \alpha$.

The floor function α is defined as

$$\alpha = \begin{cases} 0 & if\ a+b<0 \\ 1 & if\ a+b \geq 1\ with\ a \geq b \\ i & if\ a+b \geq 1\ with\ a<b \end{cases}$$

Set $a_0 = [x_0]$. Then $[x_0] = \frac{1}{x_0 - [x_0]}$ and $[x_1] = [\mathrm{Re}(x_1)] + i[\mathrm{Im}(x_1)] + \alpha$.

Again set $a_1 = [x_1]$. Then $[x_2] = \frac{1}{x_1 - [x_1]}$ and $[x_2] = [\mathrm{Re}(x_2)] + i[\mathrm{Im}(x_2)] + \alpha$.

$a_2 = [x_2]$. Then $[x_3] = \frac{1}{x_2 - [x_2]}$ and $[x_3] = [\mathrm{Re}(x_3)] + i[\mathrm{Im}(x_3)] + \alpha$ ….

$a_{k-1} = [x_{k-1}]$. Then $[x_k] = \frac{1}{x_{k-1} - [x_{k-1}]}$ and $[x_k] = [\mathrm{Re}(x_k)] + i[\mathrm{Im}(x_k)] + \alpha$.

and $a_k = [x_k]$.

The algorithm terminates if the complex continued fraction is finite.

3. FINDING COMPLEX ROOTS OF THE EQUATION $x^3 + (k-1)x^2 - (k-1)x + k = 0$ USING COMPLEX CONTINUED FRACTION:

Consider the equation $x^3 + (k-1)x^2 - (k-1)x + k = 0$. It can be factored as $(x+k)(x^2 - x + 1) = 0$

In [5] the complex roots of the quadratic equation $x^2 - x + 1 = 0$ have been found. The roots are given in terms of complex continued fraction as $\alpha_1 = \left[i,\ 2,\ \overline{-1-2i,\ 2-i}\ \right]$ and $\alpha_2 = \left[-i,\ \overline{2-i,\ -1-2i}\ \right]$

Clearly the other root of the equation is $x = -k$.

Taking $k = 1$ we now find complex roots of the equation using continued fraction method and Newton-Raphson method and compare them.

4. ILLUSTRATION:

To find the complex roots of the cubic equation $z^3 + 1 = 0$:

Method I: Using Newton-Raphson method[3,8]

Let $f(z) = z^3 + 1 = 0$

Taking $z = x + iy$ we get

$f(x+iy) = (x^3 - 3xy^2 + 1) + i(3x^2y - y^3) = 0$

Consider

$u(x,y) = (x^3 - 3xy^2 + 1) = 0$ *and* $v(x,y) = (3x^2y - y^3) = 0$

. Then

$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}$$

and $J^{-1} = \frac{1}{D}\begin{bmatrix} 3(x^2 - y^2) & 6xy \\ -6xy & 3(x^2 - y^2) \end{bmatrix}$ where

$$D = |J| = 9(x^2 - y^2)^2 + 36x^2y^2 = 9(x^2 + y^2)^2$$

Using the Newton- Raphson method $x^{(k+1)} = x^{(k)} - J_k^{-1}F_k$, we obtain

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \frac{1}{D_k}\begin{bmatrix} 3(x_k{}^2 - y_k{}^2) & 6x_k y_k \\ -6x_k y_k & 3(x_k{}^2 - y_k{}^2) \end{bmatrix}\begin{bmatrix} (x_k{}^3 - 3x_k y_k{}^2 + 1) \\ (3x_k{}^2 y_k - y_k{}^3) \end{bmatrix}$$

Taking initial approximation as $(x_0, y_0) = (0.25, 0.25)$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} - \frac{1}{0.140625}\begin{bmatrix} 0 & 0.375 \\ -0.375 & 0 \end{bmatrix}\begin{bmatrix} 0.96875 \\ 0.03125 \end{bmatrix} = \begin{bmatrix} 0.1666667 \\ 2.8333333 \end{bmatrix}$$

The successive iterations are given below

k	(x_k, y_k)
0	(0.25, 0.25)
1	(0.1666667, 2.8333333)
2	(0.15220505, 1.89374026)
3	(0.19263553, 1.27724322)

4	(0.31932197, 0.91041889)
5	(0.49252896, 0.83063199)
6	(0.49983161, 0.86738607)
7	(0.49999870, 0.86602675)

Therefore the approximation of the root is 0.5+i0.8660. The approximation of the second root is 0.5-i0.8660.

Method II: Using Continued fraction method

The number of roots of the cubic equation $z^3+1=0$ is three in which one real root -1. Then the cubic equation is reduced to the quadratic equation $z^2-z+1=0$ and it has a pair of complex roots.

In [5] we found that the complex continued fraction of the roots of the given equation $z^2-z+1=0$ are

$z_1=\lfloor i,\ 2,\ \overline{-1-2i,\ 2-i}\ \rfloor$ and $z_2=\lfloor -i,\ \overline{2-i,\ -1-2i}\ \rfloor$.

Using the properties of complex continued fraction

$p_0=a_0 \qquad p_{-1}=1$

$q_0=1 \qquad q_{-1}=0$

$p_k=a_k\,p_{k-1}+e_k\,p_{k-2}, k\geq 1$

$q_k=a_k\,q_{k-1}+e_k\,q_{k-2}, k\geq 1$

we get

$p_0=i$ and $q_0=1 \Rightarrow c_0=\dfrac{p_0}{q_0}=a_0$

$p_1=1+2i$ and $q_1=2 \Rightarrow c_1=\dfrac{p_1}{q_1}=\dfrac{1+2i}{2}=0.5+i$

$p_2=3-3i$ and

$q_2=-1-4i \Rightarrow c_2=\dfrac{p_2}{q_2}=\dfrac{3-3i}{-1-4i}=0.5+0.8824i$

$p_3=4-7i$ and

$q_3=-4-7i \Rightarrow c_3=\dfrac{p_3}{q_3}=\dfrac{4-7i}{-4-7i}=0.5+0.8615i$

$p_4=-15-4i$ and

$q_4=-11+11i \Rightarrow c_4=\dfrac{p_4}{q_4}=\dfrac{-15-4i}{-11+11i}=0.5+0.8636i$

$p_5=-30$ and

$q_5=-15+26i \Rightarrow c_5=\dfrac{p_5}{q_5}=\dfrac{-30}{-15+26i}=0.5+0.8660i$

Hence one complex root is $0.5+0.8660i$.

Obviously the pair of the above root is $0.5-0.8660i$.

From the above two methods it is observed that the number of iterations required in complex continued fraction method is fewer than the Newton- Raphson method.

5. CONCLUSION

The complex roots of the equation $x^3+(k-1)x^2-(k-1)x+k=0$ for $k\geq 1$ can be solved using Newton-Raphson method and complex continued fraction method. Factorization of this equation leads to the factors $x=-k$ and $x^2-x+1=0$. In [5] complex solutions of $x^2-x+1=0$ in terms of complex continued fraction were calculated. As a special case taking $k=1$ and comparing the solutions, we observe that the number of iterations required in complex continued fraction method is fewer than the Newton-Raphson method.

REFERENCES

[1] W. Bosma, Bastiaan cijsouw, "Complex Continued Fraction Algorithms", Thesis, Radboud University, November 2015.
[2] Ivan Niven , Herbert S. Zuckerman, Hugh L. Montgomery, "An introduction to theory of numbers", Fifth edition, Wiley Student Edition.
[3] Jain M.K. , Iyengar S.R.K., Jain R.K. " Numerical Methods for scientific and Engineering computation", Fifth edition, New Age International Publishers, New Delhi.
[4] Jonathan Browein, Alfvander Poorten, Jeffrey Shallit, Wadim Zudilin, "Neverending Fractions An Introduction to continued fractions", Cambridge University Press.
[5] Krithika S. and Gnanam A. "Analysis of complex continued fractions of the solution of the equation $kx^2-nx+n=0$ where n is an odd integer and $k=\left[\dfrac{n}{2}\right]$", International Journal of Scientific Research and Reviews, 7(4)Oct.- Dec. 2018, pp. 829-840.
[6] Neville Robbins, "Beginning Number Theory", Second edition, Narosa Publishing House.
[7] Olds, C.D., "Continued Fractions", Random House: New York, 1963.
[8] Sastry S.S., Introductory methods of Numerical Analysis , Fifth edition, PHI Private ltd. New Delhi, 2012.