Near and Closer Relations in Bitopology

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Abstract- Dvalishvili studied the concepts of near relations and closer relations in topology in 2005. In 2009, these relations were further investigated by Thamizharasi and Thangavelu. Recently the authors introduced the weak forms of near and closer relations in topology. In this paper, the weak forms of near and closer relations in bitopology have been discussed.

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1. INTRODUCTION AND PRELIMINARIES

Topologists extended the notions of semi-open, α-open, pre-open, β-open, b-open and b°-open sets in topology to bitopological spaces. The concepts of near relations and closer relations in topology that were discussed in [5, 12]. These notions are further investigated in [10]. Recently the authors introduced the weak forms of near and closer relations in topology. The purpose of this paper is to investigate these relations in bitopological settings. Throughout this paper (X,τ) is a topological space and (X, τ₁, τ₂) is a bitopological space, i,j=1,2 and i≠j. Also A and B are the subsets of X. Cl₂A = the closure of A and Int₂A = the interior of A with respect to τᵢ.

Definition 2.1: A is called
(i).regular open [9] if A= Int₂Cl₂A
(ii).semiopen[6] if there exists an open set U with U⊆ A⊆Cl₂U
(iii).preopen [7] if there exists an open set U with A⊆U⊆Cl₂A
(iv).b-open [2] if A⊆Cl₂Int₂A∪Int₂Cl₂A
(vi).b°-open [11] if A⊆Cl₂Int₂A∪Int₂Cl₂A
(vii).α-open [7] if A⊆Int₂Cl₂A.
(viii).β-open [1] if A⊆Cl₂Int₂Cl₂A.

The Complements of sets in Definition 2.1 are called the corresponding closed sets. The corresponding interior and closure operators can be defined in the usual manner. The following lemma will be useful in sequel.

Lemma 2.2:
(i).sInt₂A = A∩Cl₂Int₂A and sCl₂A = A∪Int₂Cl₂A.
(ii).pInt₂A = A∩Int₂Cl₂A and pCl₂A = A∪Int₂Cl₂A.
(iii).αInt₂A = A∩Int₂Cl₂A and αCl₂A = A∪Cl₂Int₂A.
(iv).βInt₂A = A∩Cl₂Int₂A and βCl₂A = A∪Int₂Cl₂A.

Definition 2.3: A is called
(i).ij-regular open [10] if A= Int₂Cl₂A
(ii).ij-semiopen[10] if there exists an i-open set U with U⊆ A⊆Cl₂U
(iii).ij-preopen [10] if there exists an i-open set U with A⊆U⊆Cl₂A
(iv).ij-b-open [10] if A⊆Cl₂Int₂A∪Int₂Cl₂A
(v).ij-b°-open [3] if A⊆Cl₂Int₂A∪Int₂Cl₂A
(vi).ij-α-open [10] if A⊆Int₂Cl₂Int₂A.
(vii).ij-β-open [10] if A⊆Cl₂Int₂Cl₂A.
The Complements of sets in Definition 2.3 are called the corresponding closed sets. Further A is i-regular closed if and only if \( A = \text{Cl}(\text{Int} A) \). The following lemmas have been established in [10].

**Lemma 2.5:** A is

(i) i-ji-semipreopen \( \iff \overline{A} \subseteq \text{Cl}(\text{Int} A) \).

(ii) i-ji-preopen \( \iff \overline{A} \subseteq \text{Int} \text{Cl}(A) \).

(iii) i-ji-b-closed \( \iff \text{Int}(\text{Cl}(A)) \subseteq \overline{A} \).

The concepts of sInt\( A \), and sCl\( A \) can be defined in a usual way.

**Lemma 2.6:**

(i) \( \text{sInt} A = A \cup \text{Cl}(\text{Int} A) \)

(ii) \( \text{sCl} A = A \cup \text{Int}(\text{Cl}(A)) \)

### 3. NEAR RELATIONS IN BITOPOLOGY

The next proposition shows that Lemma 2.2 can be established in bitological settings.

**Proposition 3.1:**

(i) A is i-\( \alpha \)-open iff \( A = \text{A} \cap \text{Cl}(\text{Int} A) \) and A is i-\( \alpha \)-closed iff \( A = A \cup \text{Cl}(\text{Int} A) \).

(ii) A is i-\( \beta \)-preopen iff \( A = A \cap \text{Int}(\text{Cl}(A)) \) and A is i-\( \beta \)-preclosed iff \( A = A \cup \text{Int}(\text{Cl}(A)) \).

(iii) A is i-\( \beta \)-b-closed iff \( A = A \cup \text{Cl}(\text{Int}(A)) \) and A is i-\( \beta \)-closed iff \( A = A \cup \text{Int}(\text{Cl}(A)) \).

**Proof:** Suppose A is i-\( \alpha \)-open. Then \( A \subseteq \text{Int}(\text{Cl}(A)) \) that implies \( A = A \cap \text{Cl}(\text{Int} A) \). Conversely let \( A = A \cap \text{Cl}(\text{Int} A) \) and \( A \subseteq \text{Int}(\text{Cl}(A)) \) that implies that A is i-\( \alpha \)-open. This proves the first part of (i). Now suppose A is i-\( \alpha \)-closed. Then \( A \subseteq \text{Cl}(\text{Int} A) \) that implies \( A = A \cup \text{Cl}(\text{Int} A) \). Conversely let \( A = A \cup \text{Cl}(\text{Int} A) \) and \( A \subseteq \text{Cl}(\text{Int}(A)) \) that implies A is i-\( \alpha \)-closed. This proves (ii). Other results in the proposition can be analogously established.

**Remark 3.2:** The above proposition motivates to define the closure and interior operators of a bitopological space in the following manner.

**Definition 3.3:**

(i) \( p\text{Int} A = A \cup \text{Int}(\text{Cl}(A)) \) and \( p\text{Cl} A = A \cup \text{Cl}(\text{Int}(A)) \).

(ii) \( \alpha p\text{Int} A = A \cap \text{Int}(\text{Cl}(A)) \) and \( \alpha p\text{Cl} A = A \cap \text{Cl}(\text{Int}(A)) \).

(iii) \( \beta p\text{Int} A = A \cup \text{Int}(\text{Cl}(A)) \) and \( \beta p\text{Cl} A = A \cup \text{Cl}(\text{Int}(A)) \).

**Definition 3.4:** We say that

(i) A is i-\( \alpha \)-near to B if \( \text{Int} A = \text{Int} B \)

(ii) A is i-\( \beta \)-semicnear to B if \( \text{Int} A = \text{Int} B \)

(iii) A is i-\( \beta \)-near to B if \( \alpha p\text{Int} A = \alpha p\text{Int} B \)

(iv) A is i-\( \beta \)-near to B if \( \beta p\text{Int} A = \beta p\text{Int} B \)

(v) A is i-\( \beta \)-near to B if \( \beta p\text{Int} A = \beta p\text{Int} B \)

**Proposition 3.5:**

(i) A is i-\( \alpha \)-near to B if and only if B is i-\( \alpha \)-near to A.

(ii) A is i-\( \beta \)-semicnear to B if and only if B is i-\( \beta \)-semicnear to A.

(iii) A is i-\( \beta \)-near to B if and only if B is i-\( \beta \)-near to A.

(iv) A is i-\( \beta \)-prenear to B if and only if B is i-\( \beta \)-prenear to A.

(v) A is i-\( \beta \)-near to B if and only if B is i-\( \beta \)-near to A.

**Proof:** Straight forward.

**Proposition 3.6:** Let A and B be any two subsets of X such that \( \text{Int} A \in \tau \) and \( \text{Int} B \in \tau \). If A is i-\( \beta \)-semicnear or i-\( \beta \)-near or i-\( \beta \)-near to B then A is i-\( \beta \)-near to B.

**Proof:** Suppose A is i-\( \beta \)-semicnear to B. Then \( s\text{Int} A = s\text{Int} B \) that implies \( A \cap \text{Cl}(\text{Int} A) = s\text{Int} B = B \cap \text{Cl}(\text{Int} B) \). Now \( s\text{Int}(A \subseteq s\text{Int} A = s\text{Int} B = B \cap \text{Cl}(\text{Int} B) \subseteq B \) that implies \( \text{Int} A \subseteq B \). Since \( \text{Int} A \in \tau \) it follows that \( \text{Int} A \subseteq \text{Int} B \). Similarly we can prove that \( \text{Int} B \subseteq \text{Int} A \). This proves that \( \text{Int} A = \text{Int} B \) that implies A is i-\( \beta \)-near to B. The other cases can be analogously proved.

**Proposition 3.7:**

(i) If A is i-\( \beta \)-regular then A is i-\( \beta \)-near to \( \text{Cl} A \).

(ii) If A is i-\( \beta \)-semicclosed or i-\( \beta \)-closed then A is i-\( \beta \)-near to \( \text{Cl} A \).

(iii) If A is i-\( \beta \)-preclosed then A is i-\( \beta \)-near to \( \text{Cl} \text{Int} A \).

(iv) If A is i-\( \beta \)-closed or i-\( \beta \)-b-closed or i-\( \beta \)-b*-closed then A is i-\( \beta \)-near to \( \text{Cl} \text{Int} A \).
Proof: Suppose A is ij-regular open. Then A = Int_i Cl_1 A that implies Int_1 A = Int_1 Cl_1 A. This proves that A is near to Cl_1 A in (X, τ). This proves (i). If A is ij-semi-closed then Int_1 A = Int_1 Cl_1 A that implies A is j-near to Cl_1 A. If A is ij-α-closed then Cl_1 Int_1 Cl_1 A = A that implies Int_1 A = Int_1 Cl_1 A that proves that A is j-near to Cl_1 A. This proves (ii). Suppose A is ij-precloased. Then Cl_1 Int_1 A ⊆ A that implies Int_1 A = Int_1 Cl_1 Int_1 A that proves that A is j-near to Cl_1 Int_1 A. This proves (iii). Suppose A is ij-b*-closed or ij-β-closed then A ⊇ Cl_1 Int_1 Cl_1 A that implies Int_1 A ⊇ Int_1 Cl_1 A that implies Int_1 A = Int_1 Cl_1 A that proves that A is j-near to Cl_1 Int_1 A. This proves (iv).
(iv). A is ij-pre-closer to B if and only if B is ji-α-closer to A.
(v). A is ij-β-closer to B if and only if B is ji-α-closer to A.

Proof: Straight forward.

Proposition 4.3: Let A and B be any two subsets of X such that \( X \setminus C_i(A) \in \tau_i \) and \( X \setminus C_i(B) \in \tau_i \). If A is ij-semicloser or ij-α-closer or ij-pre-closer or ij-β-closer to B then A is ij-closer to B.

Proof: Suppose A is ij-semi closer to B. Then \( sC_i(B) = sC_i(A) \) that implies \( A \cup Int_iC_iA = sC_i(A) = sC_i(B) = B \cup Int_iC_iB \). Now \( Cl_iA \supseteq Cl_iB = Cl_iA \cup B \cup Int_iC_iB \cap B \subseteq B \) that implies \( Cl_iA \subseteq Cl_iB \). Since \( X \setminus C_i(A) \in \tau_i \) and \( X \setminus C_i(B) \in \tau_i \). Similarly we can prove that \( Cl_iB \supseteq Cl_iA \). This proves that \( Cl_iA = Cl_iB \) that implies A is iij-closer to B. The other cases can be analogously established.

Proposition 4.4:
(i). If A is ij-regular closed then A is closer to Int_iA in \((X, \tau_i)\)
(ii). If A is ij-semiopen or ij-α-open then A is j-closer to Int_iA.
(iii).If A is ij-preopen or ij-b-open or b*-open then A is j-closer to Int_iCl_iA.
(iv).If A is ij-β-open then A is iij-closer to Int_iA.

Proof: Suppose A is ij-regular closed. Then A\(=Cl_iInt_iA\) that implies \( Cl_iA=Cl_iInt_iA \). This proves that A is iij-closer to Int_iA . This proves (i). Suppose A is ij-semiopen. Then \( Cl_iA=Cl_iInt_iA \) that implies A is j-closer Int_iA. Suppose A is ij-α-open. Then A\(\subseteq Int_iC_iInt_iA\) that implies \( Cl_iA=Cl_iInt_iC_iInt_iA \) that proves that A is j-closer to Int_iA. This proves (ii). Suppose A is ij-preopen. Then A\(\subseteq Int_iC_iInt_iA\) that implies \( Cl_iA=Cl_iInt_iC_iInt_iA \) that proves that A is j-closer to Int_iA. Suppose A is ij-b-open or ij-b*-open. Then A\(\subseteq Int_iC_iA\cup Cl_iInt_iA\) that implies \( Cl_iA\subseteq Int_iC_iA\cup Cl_iInt_iA \) that implies \( Cl_iA\subseteq Int_iC_iInt_iC_iA \) that implies \( Cl_iA=Cl_iInt_iC_iA \) that proves that A is j-closer to Int_iC_iA. This proves (iii). Suppose A is ij-β-open. Then A\(\subseteq C_iInt_iC_iA\) that implies \( Cl_iA=Cl_iInt_iC_iInt_iA=Cl_iInt_iC_iA \) that proves that A is iij-closer to Int_iA. This proves (iv).

Lemma 4.5: A is ij-near to B iff \( X \setminus A \) is ij-closer to \( X \setminus B \).

Proof: Since A is ij-near to B, \( Int_iA = Int_iB \) that implies \( Cl_i(X \setminus A) = Cl_i(X \setminus B) \). This proves that \( X \setminus A \) is ij-closer to \( X \setminus B \). The converse part is analogous.

Lemma 4.6: Let A be ij-closer to B and C be ij-closer to D. Then
(i). A\(\cap C \) is ij-closer to B\(\cap D \).
(ii). A\(\cap C \) is not ij-closer to B\(\cap D \).

Proof: Since A is ij-closer to B and C is ij-closer to D, \( Cl_iA=Cl_iB \) and \( Cl_iC=Cl_iD \). This implies \( Cl_iA \cup Cl_iC = Cl_i(B \cup D) \) that implies \( Cl_i(A \cup C) = Cl_i(B \cup D) \). This proves that A\(\cap C \) is ij-closer to B\(\cap D \) that implies (i). Since \( Cl_i \) \( A \cup C \neq Cl_iA \cup Cl_iC \), it follows that A\(\cap C \) is not ij-closer to B\(\cap D \).

This proves (ii).

Lemma 4.7: Let \( F_i \) be i-closed and \( F_j \) be j-closed in \((X, \tau_i, \tau_j)\). Then
(i). \( F_i \) is ij-closer to B if and only if \( F_i\subseteq Cl_iB \).
(ii). A is ij-closer to \( F_j \) if and only if \( Cl_iA = F_j \).

Proposition 4.8: Let A be ij-closer to B and B\(\subseteq A \).

Then
(i). If A is j-ij-preopen then B is j-preopen.
(ii). If B is iij-ij-ij-closed then A is iij-ij-ij-closed.
(iii). If B is ji-α-closed then A is iij-ij-ij-closed.

Proof: Suppose A is ji-α-preopen. Then \( A\subseteq Int_iC_iInt_iA \). Since A is j-ij-closer to B, \( Cl_iA=Cl_iB \) that implies \( B\subseteq A\subseteq Int_iC_iInt_iA = Int_iC_iInt_iB \). This proves that B is j-ij-preopen that proves (i).

Suppose B is ji-ij-ij-closed. Then \( B\subseteq Int_iC_iInt_iB \). Since A is ji-ij-closer to B, \( Cl_iA = C_iB \) that implies \( A\subseteq B\subseteq Int_iC_iInt_iB = Int_iC_iInt_iA \). This proves that A is iij-ij-ij-closed that proves (ii).

Suppose B is ji-α-closed. Then \( B\subseteq Cl_iInt_iC_iInt_iB \). Since A is ji-ij-closer to B \( Cl_iA = Cl_iB \) that implies \( A\subseteq B\subseteq Cl_iInt_iC_iInt_iB \subseteq Int_iC_iInt_iB = Int_iC_iInt_iA \). This proves that A is iij-ij-ij-closed that proves (iii).
5. CONCLUSION
Nearly open sets and nearly closed sets in bi topological spaces are characterized by using the near and closer relations introduced between the two topologies on a bitopological space. It has been established that every near class is a base for some topology.

REFERENCES