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Total Prime Labeling of Some Graphs

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Abstract- In this paper, we investigate the existence of prime labeling for some classes of graphs. Let G = (V,E) be a graph with ‘p’ vertices and ‘q’ edges. A labeling f: VUE $\rightarrow$ {1,2,3,……,(p+q)} is said to admit total prime labeling if for each edge e = uv, the labels assigned to u and v are relative prime, and for each vertex of degree atleast two, the greatest common divisor of the labels on the incident edges is one. A graph which admits total prime labeling is called total prime graph. We prove that the graphs Wheel (W_n), Gear (G_n), Carona ($C_{4\odot}K_5$), Triangular book (B_3^n), Splitting graph of a star ($K_{1,n}$), Double comb and Planter graph (R_n) are all total prime graphs.

Keywords: Prime labeling, vertex prime labeling, total prime labeling, splitting, planter.

1. INTRODUCTION:

Here, we consider only the graphs which are finite, simple and undirected graphs. A graph G = (V(G), E(G)) where V(G) denotes the vertex set and E(G) denotes the edge set. The order and size of the graph G are denoted by ‘p’ and ‘q’ respectively. Graph labeling where the vertices and edges are assigned real values with satisfying some conditions. Two integers are said to be relatively prime means the greatest common divisor is one.

For all other terminology and notations in graph theory, we follow Harary [1]. The notion of prime labeling was introduced by Rojer Entringer and was discussed in a paper by Tout [2] and vertex prime labeling was discussed in a paper by Deretesky [3].

Prime labeling and vertex prime labeling are already introduced. Combining these two results a new labeling called a total prime labeling was defined by Kala and Ramasubramanian [4] and they proved that the graphs P_n, cycle C_n, star $K_{1,n}$, Helm H_n and comb are total prime labeling graphs.

Definition 1.1

Let G = (V,E) be a graph with ‘p’ vertices, A labeling f: V(G) $\rightarrow$ {1,2,3,……p} is said to be as prime labeling if for each edge e = uv the labels assigned to u and v are relatively prime. A graph which admits prime labeling is called prime graph.

Definition 1.2

Let G = (V,E) be a graph with ‘p’ vertices and ‘q’ edges. A labeling f: E(G) $\rightarrow$ {1,2,……q} is said to be vertex prime labeling if for each vertex of degree atleast two the greatest common divisor of the labels on its incident edges is one.

Definition 1.3

Let G = (V, E) be a graph with ‘p’ vertices and ‘q’ edges.
A labeling f: V U E $\rightarrow$ {1,2,3,….. (p + q)} is said to be total prime labeling if,

(i) for each edge e = u v, the labels assigned to u and v are relatively prime.

(ii) for each vertex of degree atleast two, the gcd of the labels on the incident edge is one.

A graph which admits total prime labeling is called total prime graph.

Definition 1.4

The gear graph Gn is, the graph obtained from wheel W_n= C_n + K_1 by subdividing each edge incident with the apex vertex once.

Definition 1.5

The carona product of C_5 and K_5 is the graph obtained from a cycle C_n by attaching “m” new pendant edges at each vertex of cycle.

Definition 1.6

One edge union of cycles of same length is called a book. The common edge is called base of the book. If we consider n copies of cycles of length t ≥ 3, the book is denoted by B_t^n. If n = 3 the book is called triangular book graph.

Definition 1.7

A graph obtained by attaching a single pendent edge to each vertex of a path P_n = v_1,v_2,v_3…….v_n is called a comb.

Definition 1.8

For each vertex v of a graph G, take a new vertex u. Join u to all the vertices of G adjacent to v. The graph S (G) obtained is called splitting graph of G.

Definition 1.9

The planter graph R_n ($n \geq 3$) can be constructed by joining a fan graph Fn($n \geq 2$) and cycle graph C_n ($n \geq 3$) with sharing a common vertex, where n is any positive integer.

ie., $R_n = F_n + C_n$

Main Results:

Theorem 2.1: The wheel graph W_n is a total prime graph.

Proof: Let G be the wheel graph.

Let $V(W_n) = \{ v_1, v_2, v_3, \ldots\ldots v_n, v_{n+1}\}$ and

$$E(W_n) = \{ v_1 v_{i+1} \ / \ 1 \leq i \leq n \} \ U \ \{ v_i v_{i+1} \ / \ 2 \leq i \leq n \} \ U \ \{ v_{n+1} \ v_2 \}$$

The total number of vertices p is n+1 and the total number of edges q is 2n.

Here, p+q = 3n+1

Define a labeling f: VUE → { 1, 2, 3, 4, ……..(3n + 1) } by

$f(v_1) = 1$
$f(v_2) = 2$
$f(v_i) = 2i - 3$ for $3 \leq i \leq n + 1$
$f(e_i) = 2 + 2i$ for $1 \leq i \leq n - 1$
$f(ei) = n + 1 + i$;for $n \leq i \leq 2n$.

According to this pattern, the

gcd $\{f(V_1), f(V_{i+1})\}$ = gcd $\{(1, 1+i)\}$ =1, $1 \leq i \leq n$.
gcd $\{f(V_i), f(V_{i+1})\}$ = gcd $\{(i, i+1)\}$ =1, $2 \leq i \leq n$.
gcd $\{f(V_{n+1}), f(V_2)\}$ = gcd $\{(2n-1, 2)\}$ = 1.
gcd {all the edges incident with v_1 } = gcd {4,6,8,…..2n, 2n+1} = 1.
gcd {all the edges incident with v_{i+2}} = gcd {2+2i, 2n+1+i, 2n+2+i} = 1 for $1 \leq i \leq n-1$.
gcd {all the edges incident with v_2} = gcd {3n+1, 2n+1, 2n+2 } =1

Thus for each edge e = uv, where u and v are relatively prime and the gcd of each vertex of degree atleast two, all the incident edges is one.

Therefore the graph wheel W_n is a total prime graph.

Example:

Total prime labeling of W_6

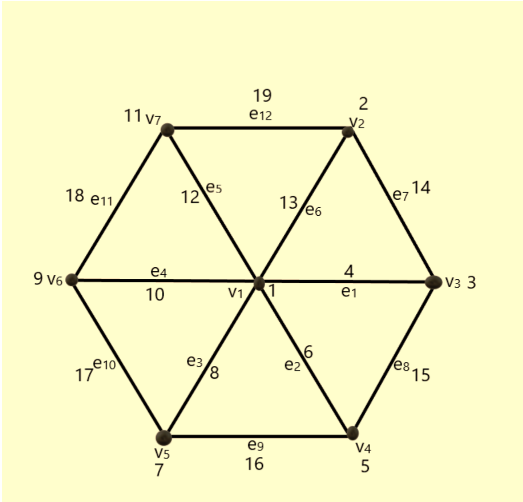


Fig.1

Theorem 2.2: The graph Gear G_n is a total prime graph.

Proof:

Let, G be the gear graph.

Let $V(G_n) = \{v_1,v_2,v_3,\ldots\ldots\ldots v_n, v_{n+1},\ldots\ldots\ldots\ldots v_{2n},v_{2n+1}\}$ and

$E(G_n) = \{v_1v_{2i} / 1 \le i\le n\}$ U $\{v_i\, v_{i+1} / 2\le i\le 2n\}$ U $\{ v_{2n+1}\, v_2\}$

The total number of vertex p is 2n+1 and the total number of edges q is 3n.
Here, p + q = 5n+1.

Define a bijection f: VUE → {1,2,3,…….(5n+1)} by
$f (v_1) = 1$
$f (v_{i+1}) = 1 + i ; 1 \le i \le 2n$
$f (e_i) = 2n + 1 + i ; 1 \le i \le 3n$

According to this pattern,

$\gcd \{ f (v_1) , f (v_{2i}) \} = \gcd \{ 1, 2i \} = 1$ for $1 \le i \le n$.
$\gcd \{ f (v_i) , f (v_{i+1}) \} = \gcd \{ i, i+1 \} = 1$ for $2 \le i \le 2n$.
$\gcd \{ f (v_{2n+1}), f (v_2)\} =$ gcd of $\{ 2n +1, 2 \} = 1$.
gcd {all the edges incident with v_1} = gcd {2n+2, 2n+3,……2n+(n+1) }
= gcd {2n+2, 2n+3,…….(3n+1)} = 1.
gcd {all the edges incident with v_{2i}, }=
gcd {2n+1+i, 3n+2i-1, 3n+2i} = 1 for $2\le i\le n$.
gcd { all the edges incident with v_2} = gcd {2n+2, 3n+2, 5n+1} = 1.
gcd { all the edges incident with v_{2i+1}} =
gcd { 3n + 2i, 3n + 2i + 1} = 1 for $1 \le i \le n$.

Therefore for each edge e = u v the gcd of {f (u), f (v)} is one, and for each vertex degree atleast two the gcd of all the incident edges is one.
Therefore, the graph Gear G_n is a total prime graph.

Example: Total prime labeling of Gear G_5

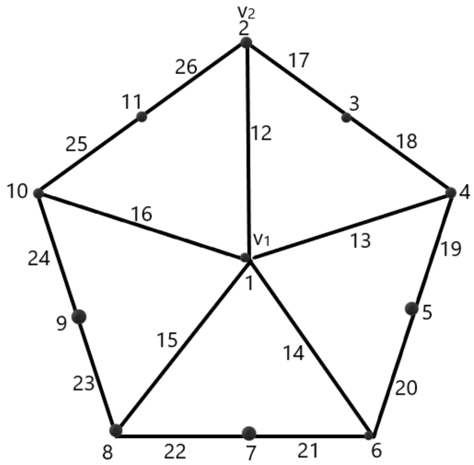


Fig.2

Theorem 2.3: The carona product $C_{4C}K_5$ is a total prime graph.

Proof:

Let G be the carona graph

$V(G) = \{ v_{ij} \ / \ 1 \leq i \leq 4 , \ 1 \leq j \leq 6 \}$

$E(G) = \{v_{ij} \ v_{ij+1} \ / \ 1 \leq i \leq 4 , \ 1 \leq j \leq 6\} \cup$
$\{ v_{11} \ v_{21} \} \cup \{ v_{21} \ v_{31} \} \cup \{ v_{31} \ v_{41} \} \cup \{ v_{41} v_{14} \}$

The total number of vertices p is n (m+1) and total number of edges q is n (m+1).
Here p+q = 2n (m+1).
Define a labeling f: V U E $\rightarrow$ { 1,2,3,……. (48) } by

$f (v_{ij}) = (5 + 1) (i – 1) + j , 1 \leq i \leq n , 1 \leq j \leq 5 + 1$
$f (e_{ij}) = (5 + 1) (4 + i – 1) + j$, where m = 5 and n = 4.

According to this pattern

$gcd \{ f (v_{11}) ,f (v_{21}) \} = gcd (1, 7) = 1$
$gcd \{ f (v_{21}), f (v_{31}) \} = gcd (7, 13) = 1$
$gcd \{ f (v_{31}), f (v_{41}) \} = gcd (13 , 19) = 1$
$gcd \{ f (v_{41}), f (v_{11}) \} = gcd (19, 1) = 1$
gcd{ all the edges incident with v_{11} } = gcd (25, 26 ,……..30) = 1
gcd{ all the edges incident with v_{21} } = gcd (31, 32, ……..36) = 1
gcd{ all the edges incident with v_{31} } = gcd (37, 38 ……..42) = 1
gcd{ all the edges incident with v_{41} } = gcd (43 ,44,……..48) = 1

Therefore, for each edge e = u v the gcd of { f(u), f(v)} is one, and for each vertex, the gcd of all the incident edges is one.

Therefore the graph carona $C_{4C}K_5$ is a total prime graph.

Example: total prime labeling of carona $C_4.K_5$.

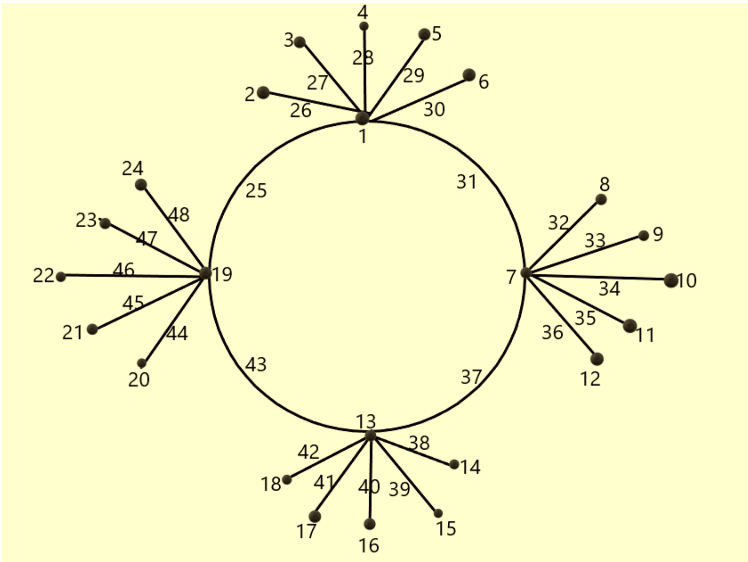


Fig.3

Theorem 2.4: The triangular book graph B^n_3 is a total prime graph if n is even.
Proof:
Let G be the triangular book graph.

Let $V(B^n_3) = \{v_1, v_2, \ldots\ldots v_n, v_{n+1}, v_{n+2}\}$ and
$E(B^n_3) = \{v_1 v_{i+1} / 1 \leq i \leq n\} \cup \{v_{i+1} v_{n+2} / 1 \leq i \leq n\} \cup \{v_1 v_{n+2}\}$

The total number of vertices p is n + 2 and total number of edges q is 2n + 1.
Hence, p + q = 3n + 3
.
Define a labeling $f: V \cup E \rightarrow \{1, 2, \ldots\ldots(3n+3)\}$ by

$$f(v_i) = \begin{cases} i; & 1 \leq i \leq m-1 \\ i+1; & m \leq i \leq n+1 \\ m; & i = n+2 \end{cases}$$
$$f(e_i) = n + 2 + i; \quad 1 \leq i \leq 2n+1.$$

Where m is the largest prime number such that $m \leq n + 2$

According to this pattern,

$\gcd\{f(v_1), f(v_{i+1})\} = \gcd \text{ of } (1, 1+i) = 1$ for $1 \leq i \leq m - 2$.
$\gcd\{f(v_1), f(v_{n+1})\} = (1, n+2) = 1$.
$\gcd\{f(v_{n+2}), f(v_{i+1})\} = \gcd \text{ of } (m, 1+i) = 1$ for $1 \leq i \leq m - 2$.
$\gcd\{f(v_{n+2}), (v_{i+1})\} = 1$ for $m \leq i \leq n + 1$.
$\gcd$ { all edges incident with v_1}
$= \gcd\{n+2+1, n+2+3, n+2+5, \ldots. n+2+(2n-1)\}$
$= \gcd\{n+3, n+5, \ldots\ldots\ldots 3n+1\} = 1$.
$\gcd$ { all edges incident with v_{n+2} }
$= \gcd\{n+3, n+2+2, n+2+4, n+2+6, \ldots n+2+2n\}$
$= \gcd\{n+3, n+4, n+6, \ldots\ldots, 3n+2\} = 1$.
$\gcd$ { all edges incident with v_{i+1} }
$= \gcd(n+2i+2, n+2i+3) =$ for $1 \leq i \leq m - 2$.
$\gcd$ {all edges incident with v_{n+1}} $= \gcd \text{ of } (3n+2, 3n+3) = 1$

Therefore, for each edge e = u v the gcd of { f (u), f (v)} is one, and for each vertex atleast degree two the gcd of all the incident edges is one.

Therefore the book graph B_3^6 is a total prime graph.

Example : Total prime labeling of book B_3^6

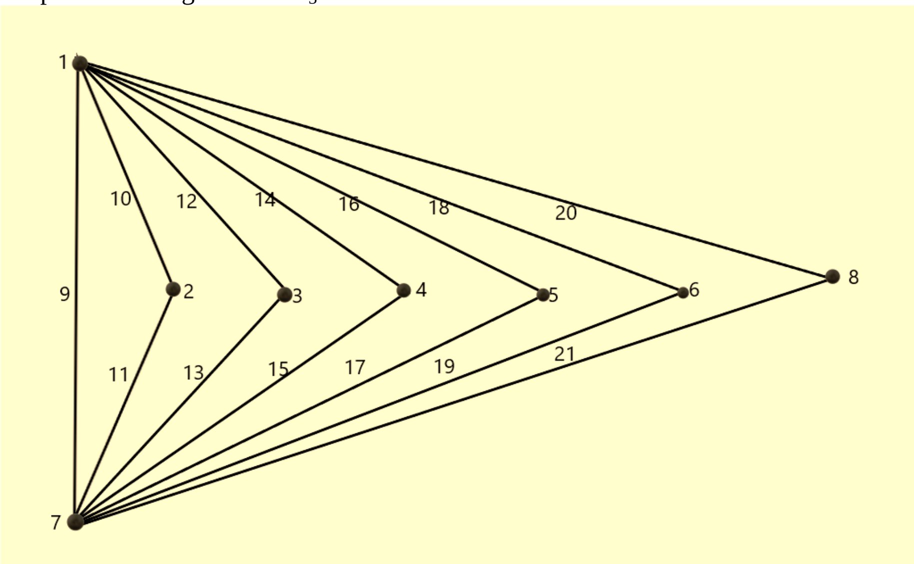


Fig.4

Theorem 2.5: Double comb $P_n.K_2$ is a total prime graph.
Proof:
Let G be a double comb graph

$V (P_n. K_{1,2}) = \{ u_1, u_2, u_3 ....u_n, v_1, v_2, v_3,........v_n, w_1, w_2,...........w_n\}$ and
$E (P_n. K_{1,2}) = \{u_i v_i / 1 \leq i \leq n \} \cup \{ v_i w_i / 1 \leq i \leq n \} \cup \{ v_i v_{i+1} / 1 \leq i \leq n - 1\}$

The total number of vertices p is 3n and the total number of edges q is 3n - 1.
Here p + q = 6n - 1.

Define a labeling f: V U E $\rightarrow$ {1,2,........(6n - 1)} by

$f (u_i) = 3i - 2; \ 1 \leq i \leq n$
$f (v_i) = 3i - 1; \ 1 \leq i \leq n$
$f (w_i) = 3i; \ 1 \leq i \leq n$
$f (e_i) = 3n + i; 1 \leq i \leq 3n - 1$

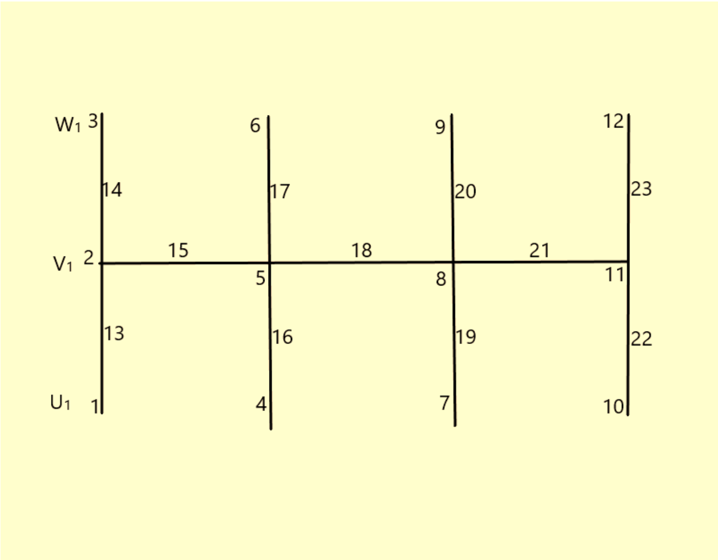
According to this pattern,

gcd { $f (u_i), f (v_i)$ } = gcd of {3i - 2, 3i - 1} = 1 for $1 \leq i \leq n$.
gcd { $f (v_i), f (w_i)$} = gcd of {3i - 1, 3i } = 1 for $1 \leq i \leq n$.
gcd{ $f(v_i), f (v_{i+1})$} = gcd of {3i - 1, 3i + 2} = 1 for $1 \leq i \leq n - 1$.
gcd{ all the edges incident with v_1} = gcd of {3n + 1, 3n + 2, 3n + 3 } = 1
gcd {all the edges incident with v_n} = gcd of {6n - 3, 6n - 2, 6n - 1} = 1
gcd {all the edges incident with v_i / $2 \leq i \leq n - 1$}
= gcd {incomplete}

Therefore, for each edge e = u v the gcd of { f (u), f (v)} is one, and for each vertex degree atleast two the gcd of all the incident edges is one.

Therefore P_n. K_2 is a total prime graph.

Example : Total prime graph of (P_4. $K_{1,2}$)



Theorem 2.6: Splitting graph of a star graph S ($K_{1,n}$) is a total prime graph.

Proof: Let the vertices and edges of a star graph is

V (G) = {v_1, v_2, v_3,.........v_{n+1}} and
E (G) = { $v_1 v_{i+1}$ / $1 \leq i \leq n$ }

Let S (G) be a splitting graph of $K_{l,n}$ and {u_1, u_2, ...u_n, u_{n+1}} be the newly added vertices with $K_{l,n}$
Now, V { S ($K_{l,n}$) } = { v_1, v_2,.........v_n, v_{n+1}, u_1, u_2,..........u_n, u_{n+1}} and
E {S ($K_{1,n}$)} = {v v_i / $1 \leq i \leq n$ } U { v_i u / $1 \leq i \leq n$ } U { u u_i / $1 \leq i \leq n$ }

Hence, the total number of vertices p = 2n + 2 and total number of edges q = 3n.
Here p + q = 5n + 2.

Define a labeling f: V U E → {1,2,.......(5n+2)}by

f (v) = 2
f (u) = 1
f (u_i) = 2i; $1 \leq i \leq n$
f (v_i) = 2i + 1; $1 \leq i \leq n$
f (e_i) = 2n + 2 + i; $1 \leq i \leq 3n$.

According to this pattern

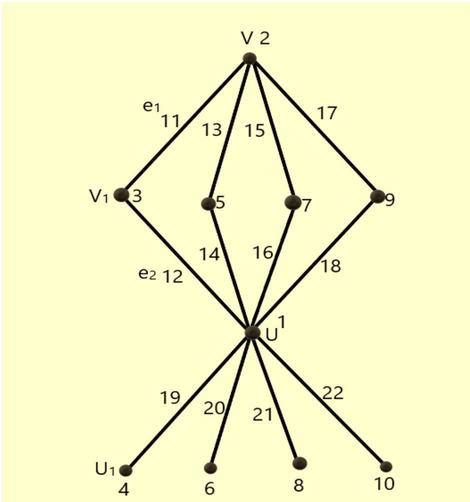
gcd {all the edges incident with v}
= gcd {2n + 2 + 1, 2n + 2 + 3, 2n + 2 + 5,....2n + 2 + (2n – 1) }
=gcd {2n + 3,2n + 5,……4n + 1} = 1.
gcd {all the edges incident with u}
= gcd {2n + 2 + 2, 2n + 2 + 4,......,2n + 2 + 2n, 4n + 2 + 1,
4n + 2 + 2,.....4n + 2 + n }
= gcd {2n + 4, 2n + 6,.......4n + 2, 4n + 3, 4n + 4,.........5n + 2 } = 1.
gcd {all the edges incident with vi}

= gcd {2n + 1 + 2i, 2n + 2 + 2i} = 1 for $1 \leq i \leq n$
gcd {f (v), f (v_i)} = gcd {2, 2i + 1} = 1; $1 \leq i \leq n$
gcd {f (v_i), f (u) } = gcd {2i + 1, 1} = 1; $1 \leq i \leq n$
gcd {f (u), f (u_i)} = gcd {1, 2i} = 1; $1 \leq i \leq n$

Therefore, for each edge e = u v the gcd of { f (u), f (v) is one and for each vertex, the gcd of all the incident edges is one. Therefore the splitting graph of a star graph S ($K_{1,n}$) is a total prime graph.

Example:

Total prime graph of splitting graph of a star graph S ($K_{1,4}$)



Theorem 2.7: The planter graph R_n ($n \geq 3$) is total prime graph.

Proof:

Let G be a planter graph (R_n)

V (R_n) = {v_1, v_2,.........v_n, v_{n+1},........v_{2n} }
E (R_n) = {$v_1 v_{i+1}$ / $1 \leq i \leq n$} U {$v_i v_{n+2}$ } U {$v_{n+1+i} v_{n+2+i}$ / $1 \leq i \leq n - 2$ } U {$v_{2n} v_1$}.

The total number of vertices p is 2n and total number of edges q is 3n - 1. p + q = 5n - 1.

Define a labeling f: V U E → {1,2,....(5n-1)} by

f (v_i) = i; $1 \leq i \leq 2n$
f (e_i) = 2n + i; $1 \leq i \leq 3n - 1$.

According to this pattern

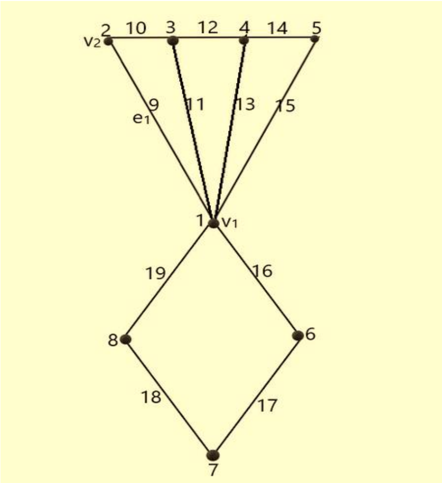
gcd { f (v_1), f (v_{i+1}) } = gcd {1, 1 + i} = 1 for $1 \leq i \leq n$.
gcd {f (v_1), f (v_{n+2})} = gcd { 1, n + 2 } = 1.
gcd { f (v_{n+1+i}), f (v_{n+2+i})} = gcd {n +1 + i, n + 2 + i} = 1 for $1 \leq i \leq n - 2$.
gcd { f (v_{2n}), f (v_1)} = gcd {2n,1} = 1.
gcd { f (v_{i+1}),f (v_{i+2})} = gcd {1 + i,2 + i} = 1, $1 \leq i \leq n - 1$
gcd {all the edges incident with v_1}
= gcd {2n + 1, 2n + 3 2n + 5, 2n + 7,......2n + 2n -1, 2n + 2n -1 + 1, (5n - 1)}
= gcd {2n +1, 2n +3, 2n + 5,.....(4n - 1), 4n, (5n -1) } = 1.
gcd {all the edges incident with v_2} = gcd {2n +1,2n + 2} = 1.

gcd {all the edges incident with v_{n+1}} = gcd {4n - 2, 4n -1} = 1.
gcd {all the edges incident with (v_{n+1+i}) }
= gcd { 4n -1 + i,4n + i}=1 for $1 \leq i \leq n -1$
gcd {all the edges incident with v_{i+2}}=gcd (incomplete)

Therefore, for each edge e = u v the gcd of {f (u), f (v)} is one and for each vertex, the gcd of all the incident edges is one. Therefore the planter graph (R_n) is a total prime graph.

Example:

Total prime graph of planter graph (R_4)



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