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On The Diophantine Equation $2^x + 7^y = z^2$

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Abstract- In this paper, we show that the Diophantine equation $2^x + 7^y = z^2$ has exactly three solutions in non-negative integers for x, y and z. The solutions are $(3, 0, 3)$, $(5, 2, 9)$ and $(1, 1, 3)$.

Keywords- Exponential equation**;** Diophantine equation; integer solution**.**

1. INTRODUCTION

There are lots of studies about the Diophantine equation of the type $a^x + b^y = c^z$. In 1844, Catalan conjectured that (3, 2, 2, 3) is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with *min* {a, b, x, y} > 1. This conjecture was proven by Mihailescu in 2004. In 2005, D. Acu [1] showed that the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions in non-negative integers i.e. (x, y, z) ∈ {(3, 0, 3), (2, 1, 3)}. In 2013, Chotchaisthit [2] studied the Diophantine equation $2^x + 11^y = z^2$ and found that (3, 0, 3) is a unique non-negative integer solution. In the same year, B. Sroysang [3] solved the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. He showed that this equation has three solutions in $(x, y, z) \in$ {(0,1,2),(3,0,3),(4,2,5)}. Again in the same year, the same author [4] solved the Diophantine equation $2^x + 19^y = z^2$ and showed that it has unique non-negative integer solution (3, 0, 3) in x, y and z. In 2013, Rabago [5] found that the Diophantine equation $2^x + 31^y = z^2$ has exactly two solutions in non-negative integers. The solutions (x, y, z) are $(3, 0, 3)$ and $(7, 2, 33)$. In 2016, Rabago [6] showed that the Diophantine equation $2^x + 17^y = z^2$ has exactly five solutions (x, y, z) in positive integers. The only solutions are $(3, 1, 5), (5, 1, 7), (6, 1, 9), (7, 3, 71)$ and $(9, 1, 23)$. In 1958, Nagell [7] gave solutions of several Diophantine equations of type $a^x = b^y + c^z$ in which 7 with a power was one of the components $7^x = p^y + 2^z$, $3^x = 7^y + 2^z$, $7^x = 3^y + 2^z$, $2^x = 7^y + 3^z$, $5^x = 7^y + 2^z$, $2^x = 7^y + 5^z$, $7^x = 5^y + 2^z$. For related papers, we refer to [8, 9, 10, 11, 12, 13, 15, 16, 17]. In this short communication, as inspired by these aforementioned works, we show that the equation $2^x + 7^y = z^2$ has exactly three solutions in non-negative integers (x, y, z).

2. PRELIMINARIES

We start this section by presenting a Proposition and a lemma.

2.1. ***Proposition [14]***

The Catalan's Conjecture states that $(3, 2, 2, 3)$ is a solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [1]

$(3, 3)$ is a unique solution (x, y) for the Diophantine equation $2^x + 1 = z^2$ where x and z are non-negative integers.

3. MAIN RESULT

In this section we present our main theorem.

Theorem 3.1. The Diophantine equation

$$2^x + 7^y = z^2 \quad \ldots\ldots\ldots\ldots..(1)$$

has exactly three solutions in non-negative integers $(x, y, z) \in \{(3, 0, 3), (5, 2, 9), (1, 1, 3)\}$.

Proof. Let x, y and z be non-negative integers such that $2^x + 7^y = z^2$. We first consider the case when y is zero. By lemma 2.2, we have $(x, y, z) = (3, 0, 3)$. Now we consider the case when $x, y, z > 0$. Here two cases arise:

Case (I) If y is even i.e. $y = 2k$, where k is a natural number then we have

$$2^x + 7^{2k} = z^2$$

Or, $$2^x = (z - 7^k)(z + 7^k)$$

where $(z - 7^k) = 2^u$ and $(z + 7^k) = 2^{x-u}$, u is a natural number. Here we obtain

$$2 \cdot 7^k = 2^u(2^{\{x-2u\}} - 1)$$

which implies that $u = 1$ and $2^{x-2u} - 1 = 7^k$. Then $2^{\{x-2\}} - 7^k = 1$. Since $k \geq 1$, we obtain that $2^{\{x-2\}} = 7^k + 1 \geq 7^1 + 1 = 8$. Thus $x \geq 5$. By Proposition 2.1, we have $k = 1$. Thus $2^{x-2} = 8 = 2^3$. Hence, $x = 5$.

For $k = 1$ and $x = 5$, we have $y = 2$ and $z = 9$. Therefore, the solution is $(x, y, z) = (5, 2, 9)$.

Case (II) If y is odd i.e., $y = 2k + 1$, k is a non-negative integer, then Equation (1) can be written as

$$2^x + 7^{2k+1} = z^2$$

or,

$$2^x + (4 + 3) \cdot 7^{2k} = z^2$$

$$2^x + 3 \cdot 7^{2k} = z^2 - 4 \cdot 7^{2k}$$

which implies that
$2^x + 3 \cdot 7^{2k} = (z - 2 \cdot 7^k)\,(z + 2 \cdot 7^k)$
Or, $(z - 2 \cdot 7^k)(z + 2 \cdot 7^k) = 2^x + 3 \cdot 7^{2k}$
So we have the following equality.

$$z + 2 \cdot 7^k = 2^x + 3 \cdot 7^{2k} \ldots\ldots\ldots (i)$$
$$z - 2 \cdot 7^k = 1 \ldots\ldots\ldots (ii)$$

Eliminating z from above equation, we have

$$7^k(4 - 3 \cdot 7^k) = 2^x - 1$$

This implies $k = 0$ and $4 - 3 \cdot 7^k = 2^x - 1$. Then $4 - 3 = 2^x - 1$ which implies $x = 1$. For $k = 0$ and $x = 1$, we have $y = 1$ and $z = 3$. Hence the solution is $(x, y, z) = (1, 1, 3)$. This completes the proof of the theorem.

Corollary 3.2. The Diophantine equation $2^x + 7^y = w^4$ has a unique positive integer solution in $(x,\ y,\ w)$ as $(5, 2, 3)$.

Proof. Suppose that there are positive integers x, y and w such that $2^x + 7^y = w^4$. Let $z = w^2$. Then $2^x + 7^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (5, 2, 9)$. Thus, $z = w^2 = 9$. So $w = 3$. Therefore, the Diophantine equation $2^x + 7^y = w^4$ has a unique positive integer solution in $(x,\ y,\ w)$ as $(5,\ 2,\ 3)$.

4. CONCLUSION

In this paper, we have solved the Diophantine equation $2^x + 7^y = z^2$ where 7 is prime. We have shown that the entitled equation has three non-negative integer solutions in $(x,\ y,\ z) \in \{(3,\ 0,\ 3),\ (5,\ 2,\ 9),\ (1,\ 1,\ 3)\}$.

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