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On The Cubic Equation with Four Unknowns $x^3 + 25z^3 = y^3 + 25w^3 + 21(x - y)^3$

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Abstract: The sequences of integral solutions to the cubic equation with four variables are obtained. A few properties among the solutions are also presented.

Index Terms: Cubic; Cubic with four unknowns; integer solutions

1. INTRODUCTION

The Diophantine equation offer an unlimited field for research due to their variety [1,2]. In particular, one may refer [3-13] for cubic equation with three unknowns. In [14-18] cubic equation with four unknowns are studied for its non-trivial integral solutions. This communication concerns with the problem of obtaining non-zero integral solution of cubic with four variables is given by $x^3 + 25z^3 = y^3 + 25w^3 + 21(x - y)^3$. A few properties among the solutions and special numbers are presented.

Notations

- $ct_{m,n} = \frac{mn(n+1)+2}{2}$
- $t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$
- $P_n^m = \frac{n(n+1)}{6}[(m-2)n + (5-m)]$
- $SO_n = n(2n^2 - 1)$
- $s_n = 6n(n-1) + 1$
- $Pr_n = n(n+1)$
- $Ky_n = 2^{2n} + 2^{n+1} - 1$
- $Gno_n = 2n - 1$
- $CS_n = n^2 + (n-1)^2$
- $PP_n = \frac{1}{2}n^2(n+1)$
- $Tha_n = 3(2)^n - 1$
- $FN_n^4 = \frac{n^2(n^2-1)}{12}$
- $j_n = 2^n + (-1)^n$
- $J_n = \frac{1}{3}[2^n + (-1)^n]$
- $Sqp_k = \frac{n(n+1)(2n+1)}{6}$

2. METHOD OF ANALYSIS

The cubic Diophantine equation with four unknowns to be solved for getting non-zero integral solution is

$$x^3 + 25z^3 = y^3 + 25w^3 + 21(x - y)^3 \quad (1)$$

On substituting the linear transformation

$$x = u + v,\ y = u - v,\ w = s + v,\ z = s - v \quad (2)$$

in (1) leads to

$$u^2 = 36v^2 + 25s^2 \quad (3)$$

We solve (3) in different ways and we get different pattern of integral solutions to (1).

2.1. *Pattern I*

In equation (3), which is satisfied by,

$$v = \frac{1}{3}pq$$

$$s = \frac{p^2 - q^2}{5}$$

$$u = p^2 + q^2$$

Put p=15p, q=15q in v, s, u we get,

$$v = 75q$$

$$s = 45(p^2 - q^2)$$

$$u = 225(p^2 + q^2)$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(p,q) = 225(p^2+q^2)+75pq$$

$$y(p,q) = 225(p^2+q^2)-75pq$$

$$w(p,q) = 45(p^2-q^2)+75pq$$

$$z(p,q) = 45(p^2-q^2)-75pq$$

2.1.1. *Properties*

1. $6\left(\frac{x(p,p+1)+y(p,p+1)-450}{Pr_p}\right)$ is a Nasty Number.
2. $\frac{w(p,p+1)-z(p,p+1)}{Pr_p}$ is a Nasty Number.
3. $w(p,t_{4,p})-z(p,t_{4,p})+33SO_p+33t_{2,p}$ is a cubical integer.
4. $w(p,p+1)+z(p,p+1)-90Gno_p=0$
5. $x(p+1,t_{4,p})-y(p+1,t_{4,p})=300P_p^5$
6. $\frac{w(p,p+1)+z(p,p+1)}{90}\equiv 1(mod\ 2)$
7. $x(p,p+1)+z(p,p+1)-y(p,p+1)-w(p,p+1)=0$

2.2. *Pattern II*

Equation (3) can be re-written as

$$u^2-(6v)^2=(5s)^2$$

Which is written in the form of ratio as,

$$\frac{u+6v}{5s}=\frac{5s}{u-6v}=\frac{A}{B}\ \ where\ B\neq 0 \tag{4}$$

Which is equivalent to the system of equations,

$$Bu+6Bv-5As=0$$

$$-Au+6Av+5Bs=0$$

Applying the method of cross multiplication, we have

$$u=30A^2+30B^2$$

$$v=5A^2-5B^2$$

$$s=12AB$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(A,B)=35A^2+25B^2$$

$$y(A,B)=25A^2+35B^2$$

$$w(A,B)=5A^2-5B^2+12AB$$

$$z(A,B)=5B^2-5A^2+12AB$$

2.2.1. *Properties*

1. $w(A+1,t_{4,A})+z(A+1,t_{4,A})=48P_A^5$
2. $w(A,A-1)+z(A,A-1)+4=4S_A$
3. $x(A,A-1)-y(A,A-1)+w(A,A-1)-z(A,A-1)=20Gno_A$
4. $y(A,A-1)+w(A,A-1)+12Pr_A=30CS_A+24t_{4,a}$
5. $x((A,A)+y(A,A)+w(A,A)+z(A,A)$ is a perfect square.
6. $w(A,t_{3,a})+z(A,t_{3,a})=24P_A^5$
7. $x(A,A-1)+y(A,A-1)+w(A,A-1)-132t_{4,a}\equiv 55(mod\ 122)$

2.3. *Pattern III*

Equation (4) can also be written as

$$\frac{u+6v}{25s}=\frac{s}{u-6v}=\frac{A}{B}\ \ where\ B\neq 0$$

Which is equivalent to the system of equations,

$$Bu+6Bv-25As=0$$

$$-Au+6Av+Bs=0$$

Applying the method of cross multiplication, we have

$$u=150A^2+6B^2$$

$$v=25A^2-B^2$$

$$s=12AB$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(A,B)=175A^2+5B^2$$

$$y(A,B)=125A^2+7B^2$$

$$w(A,B)=25A^2-B^2+12AB$$

$$z(A,B)=B^2-25A^2+12AB$$

2.3.1. *Properties*

1. $x(2^n,1)+y(2^n,1)-312=300j_{2n}$
2. $x(2^n+1,1)-y(2^n+1,1)=50Ky_n+98$
3. $x(1,1)-y(1,1)+w(1,1)-z(1,1)$ is a Nasty Number.
4. $x(A,1)-y(A,1)-w(A,1)+6Gno_A+j_3$ is a perfect square.
5. $y(A,A+1)-4w(A,A+1)+z(A,A+1)+\ 4S_A+18Gno_A+2=0$
6. $x(1,B)+y(1,B)-4w(1,B)+24Gno_B-176$ is a perfect square.
7. $x(1,t_{4,p})+y(1,t_{4,p})+w(1,t_{4,p})+z(1,t_{4,p})-\ 144FN_p^4-36t_{4,p}=300$

2.4. *Pattern IV*

Equation (4) can also be written as

$$\frac{u+6v}{s}=\frac{25s}{u-6v}=\frac{A}{B}\ \ where\ B\neq 0$$

Which is equivalent to the system of equations,

$$Bu+6Bv-As=0$$

$$-Au+6Av+25Bs=0$$

Applying the method of cross multiplication, we have

$$u=6A^2+150B^2$$

$$v=A^2-25B^2$$

$$s=12AB$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(A,B)=7A^2+125B^2$$

$$y(A,B)=5A^2+175B^2$$

$$w(A,B)=A^2-25B^2+12AB$$

$$z(A,B)=25B^2-A^2+12AB$$

2.4.1 *Properties*

1. $w(A,t_{3,A})+z(A,t_{3,A})=24P_A^5$
2. $x(2^n,1)+y(2^n,1)-288=12j_{2n}$
3. $x(A,A+1)-y(A,A+1)+48Pr_A+26Gno_A$ is a Nasty Number.
4. $w(A,A+1)+z(A,A+1)=24Pr_A$
5. $x(B+1,B)-y(B+1,B)+48Pr_B\equiv -2(mod\ 52)$
6. $w(A,A)+z(A,A)+Pr_A-t_{2,A}$ is a perfect square.

2.5. *Pattern V*

Equation (3) can be re-written as

$$u^2-(5s)^2=(6v)^2$$

Which is written in the form of ratio as,

$$\frac{u+5s}{6v}=\frac{6v}{u-5s}=\frac{A}{B}\ \ where\ B\neq 0 \qquad (5)$$

Which is equivalent to the system of equations,

$$Bu-6Av+5Bs=0$$

$$-Au+6Bv+5As=0$$

Applying the method of cross multiplication, we have

$$u=30A^2+30B^2$$

$$v=10AB$$

$$s=6A^2-6B^2$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(A,B)=30A^2+30B^2+10AB$$
$$y(A,B)=30A^2+30B^2-10AB$$
$$w(A,B)=6A^2-6B^2+10AB$$
$$z(A,B)=6A^2-6B^2-10AB$$

2.5.1 *Properties*

1. $x(2^{2A-1},1)+y(2^{2A-1},1)=60j_{4A-2}$
2. $x(2^n,1)+y(2^n,1)-w(2^n,1)+z(2^n,1)-120=\ 180J_{2n}-30J_n-10j_n$
3. $x(A,A)+y(A,A)+t_{4,A}$ is a perfect square.
4. $y(A,1)+w(A,1)-24$ is a perfect square.
5. $x(1,t_{3,A})-y(1,t_{3,A})+w(1,t_{3,A})-z(1,t_{3,A})+\ 4Pr_A$ is a Nasty Number.
6. $x(A,A+1)-y(A,A+1)=20Pr_A$
7. $y(A,1)+w(A,1)-36Pr_A+36t_{2,A}$ is a Nasty Number.

2.6. *Pattern VI*

Equation (5) can also be written as

$$\frac{u+5s}{v}=\frac{36v}{u-5s}=\frac{A}{B}\ \ where\ B\neq 0$$

Which is equivalent to the system of equations,

$$Bu-Av+5Bs=0$$

$$-Au+36Bv+5As=0$$

Applying the method of cross multiplication, we have

$$u=5A^2+180B^2$$

$$v=10AB$$

$$s=A^2-36B^2$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(A,B)=5A^2+180B^2+10AB$$

$$y(A,B)=5A^2+180B^2-10AB$$

$$w(A,B)=A^2-36B^2+10AB$$

$$z(A,B)=A^2-36B^2-10AB$$

2.6.1 *Properties*

1. $x(A,A+1)-y(A,A+1)=20Pr_A$
2. $30P_A^5\left[w\left(A,t_{3,A}\right)-z\left(A,t_{3,A}\right)\right]$ is a Nasty Number.
3. $w(2^n,1)+z(2^n,1)+72=3J_{2n}+j_{2n}$
4. $x(A,A)-w(A,A)+36Pr_A-18Gno_A-18$ is a perfect square.
5. $y(A,A+1)+z(A,A+1)-130Pr_A-69Gno_A-213=0$
6. $x(B+1,B)+y(B+1,B)+w(B+1,B)+z(B+1,B)-12t_{4,B+1}=288t_{4,B}$
7. $x(A,A)+w(A,A)-t_{4,A}$ is a perfect square.

2.7. *Pattern VII*

Equation (5) can also be written as

$$\frac{u+5s}{36v}=\frac{v}{u-5s}=\frac{A}{B}\ \ where\ B\neq 0$$

Which is equivalent to the system of equations,

$$Bu-36Av+5Bs=0$$

$$-Au+Bv+5As=0$$

Applying the method of cross multiplication, we have

$$u=180A^2+5B^2$$

$$v=10AB$$

$$s=36A^2-B^2$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x(A,B)=180A^2+5B^2+10AB$$

$$y(A,B)=180A^2+5B^2-10AB$$

$$w(A,B)=36A^2-B^2+10AB$$

$$z(A,B)=36A^2-B^2-10AB$$

2.7.1 *Properties*

1. $x(2^n,1)+y(2^n,1)+w(2^n,1)+z(2^n,1)-\ 440=1296J_{2n}$
2. $x(2^n,1)+y(2^n,1)-w(2^n,1)-z(2^n,1)-108=96Tha_{2n}$
3. $y(A,1)-w(A,1)+10Gno_A+4$ is a perfect square.
4. $y(A,1)-w(A,1)+6Pr_A+7Gno_A+j_1$ is a Nasty Number.
5. $x(B+1,B)-z(B+1,B)-170Pr_B-69Gno_B=213$
6. $w(A,A)-t_{92,A}\equiv 0(mod\ 44)$
7. $x(A,A+1)-z(A,A+1)-85CS_A\equiv 79(mod\ 202)$

2.8. *Pattern VIII*

Equation (3) can be re-written as

$$u^2-(5s)^2=36v^2$$

Which is equivalent to the system of equations,

$$u+5s=36v$$

$$u-5s=v \quad (6)$$

Solving these two linear equations, we have

$$u = \frac{37v}{2}$$

$$s = \frac{7v}{2}$$

Put $v = 2k$ then we get the integer solution of u and s are as

$$u = 37k$$

$$s = 7k$$

Substituting the values of u, v, s in (2), we get the non-trivial integer solutions of equation (1) are given by

$$x = 39k$$

$$y = 35k$$

$$w = 9k$$

$$z = 5k$$

2.9. *Pattern IX*

The system of equation (6) can be written as

$$u + 5s = v^2$$

$$u - 5s = 36$$

Solving these two equations, we have

$$u = \frac{v^2 + 36}{2}$$

$$s = \frac{v^2 - 36}{10}$$

Taking $v = k^2 + k + 6$ in the above equation, we get

$$u = \frac{(k^2 + k)^2}{2} + 6(k^2 + k) + 36$$

$$s = \frac{(k^2 + k)(k^2 + k + 6) + 6(k^2 + k)}{10}$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x = \frac{6k^4 + 12k^3 + 78k^2 + 72k + 350}{10}$$

$$y = \frac{4k^4 + 8k^3 + 52k^2 + 48k + 350}{10}$$

$$w = \frac{k^4 + 2k^3 + 23k^2 + 22k + 60}{10}$$

$$z = \frac{k^4 + 2k^3 + 3k^2 + 2k - 60}{10}$$

2.9.1 *Properties*

1. $w - z = 2Pr_k + 12$
2. $4w - y + 9 = 2CS_k + 4Gno_k$
3. $10(x + w) - 84FN_k^4 - 42Sqp_k = 87Pr_k + 410$
4. $x + y - 12FN_k^4 - 4PP_k = Ct_{24,k} + 69$
5. $10(x - y) - 24FN_k^4 - 8PP_k = 48t_{3,k}$
6. $4w - y = Ct_{8,k} - 12$

2.10. *Pattern X*

Equation (3) can be re-written as

$$u^2 - (6v)^2 = 25s^2$$

Which is equivalent to the system of equations,

$$u + 6v = s^2$$

$$u - 6v = 25$$

Solving these two equations, we have

$$u = \frac{s^2 + 25}{2}$$

$$v = \frac{s^2 - 25}{12}$$

Taking $v = k^2 + k + 5$ in the above equation, we get

$$u = \frac{(k^2 + k)^2}{2} + 5(k^2 + k) + 25$$

$$v = \frac{(k^2 + k)(k^2 + k + 5) + 5(k^2 + k)}{12}$$

In view of (2) the non-zero integer solutions to (1) are given by

$$x = \frac{7k^4 + 14k^3 + 77k^2 + 70k + 300}{12}$$

$$y=\frac{5k^4+10k^3+55k^2+50k+300}{12}$$

$$w=\frac{k^4+2k^3+23k^2+22k+60}{12}$$

$$z=\frac{-k^4-2k^3+k^2+2k+60}{12}$$

2.10.1. *Properties*

1. $z+w=4t_{3,k}+10$
2. $2z+2w-19=Ct_{8,k}$
3. $5w-y=60Pr_k$
4. $7y-5x=50$
5. $y+5z=10t_{3,k}+50$
6. $5w-y=10t_{3,k}$
7. $7w-x=Ct_{14,k}+9$

3. CONCLUSION

One may search for other pattern of solutions and their corresponding properties.

REFERENCES

[1] Dickson. L.E, "History of the theory of numbers", Vol.2: Diophantine Analysis, Newyork: Dover, 2005.

[2] Carmichael. R. D., "The theory of numbers and Diophantine Analysis", New York: Dover, 1959.

[3] Gopalan. M.A, Manju somnath and vanitha.N "On Ternary Cubic Diophantine Equation $x^2+y^2=2z^3$ ",Advances in Theoretical and Applied Mathematics Vol. 1,No.3 pp. 227-231, 2006.

[4] Gopalan. M.A, Manju Somnath and Vanitha.N "On Ternary Cubic Diophantine Equation $x^2+y^2=z^3$" Acta Ciencia Indica,Vol,XXXIIIM,No.3.pp.705-707,2007.

[5] Gopalan,M.A., and Anbuselvi.R "Integral solution of ternary cubic Diophantine equation "On Ternary Cubic Diophantine Equation $x^2+y^2+4N=zxy$", pure and Applied Mathematics Sciences, Vol.LXVII,No.1-2, March pp.107-111,2008.

[6] Gopalan. M.A, Manju Somnath and Vanitha.N "Note on the equation $x^3+y^3=a(x^2-y^2)+b(x+y)$",International Journal of Mathematics, Computer Sciences and Information Technologies Vol.No-1, January-June, pp 135-136, 2008.

[7] Gopalan M.A, and Pandichelvi.V, "Integral Solutions of Ternary Cubic Equation $x^2-xy+y^2=(k^2-2k+4)z^3$" ,Pacific-Asian Journal of Mathematics Vol 2, No1-2,91-96,2008.

[8] Gopalan M.A , Devibala. S and Manju Somnath, "Integral solutions of $x^3+x+y^3+y=4(z-2)(z+2)$ "Impact J.Sci.Tech., Vol.2(2) pp 65-69,2008.

[9] Gopalan. M. A, Manju Somnath and Vanitha.N, "On Ternary Cubic Diophantine Equation $2^{2\alpha-1}(x^2+y^2)=z^3$, Acta Ciencia Indica, Vol,XXXIVM, No.3, pp.135-137,2008.

[10] Gopalan. M. A, KalingaRani.J "Integral solutions of $x^3+y^3+8k(x+y)=(2k+1)z^3$, "Bulletin of pure and Applied Sciences,Vol.29E,(No.1)pp95-99,2010.

[11] Gopalan. M. A and Janaki.G "Integral solutions of $x^2-y^2+xy=(m^2-5n^2)z^3$, Antartica J. Math.,7(1)pg.63-67,2010.

[12] Gopalan. M. A and sanmuganatham. P "On the Equation $x^2+xy-y^2=(n^2+4n-1)z^3$ "Bulletin of pure and Applied Sciences,Vol.29E, Pg 231-235 Issue 2, 2010.

[13] Gopalan. M. A and vijayasankar. A "Integral Solutions oh Ternary Cubic Equation $x^2+y^2-xy+2(x+y+2)=z^3$ "Antartica J.Math., Vol.7(No.4)pg.455-460,2010.

[14] Gopalan. M. A,Vidhyalakshmi, and G. Sumathi "On the homogeneous cubic equation with four unknowns $x^3+y^3=14z^3-3w^2(x+y)$, Discovery J. Math Vol.2,No.4,pg 17-19,2012.

[15] Gopalan. M. A, Sumathi and Vidhyalakshmi "On the homogeneous cubic equation with four unknowns $x^3+y^3=z^3+w^2(x+y)$, Diophantus J. Math, Vol 2(2), pg 99-103,2013.

[16] Gopalan. M. A, S. Vidhyalakshmi and T. R. Usha Rani "On the homogeneous cubic equation with four unknowns $x^3+y^3=(z+w)^2(z+w)$" Sci. J.Eng.Tech.,2014:2(2B):264-266.

[17] Gopalan. M. A, S. Vidhyalakshi, T. R. Usha Rani and v. Karthika "On the cubic equation with four unknowns $x^3+y^3=14zw^2$" ijsrp, Vol-5, Issue-3,March-2015.

[18] Gopalan. M. A,Vidhyalakshmi. S and Santhi.J "On the Cubic Equation with four unknowns $x^3+4z^3=y^3+4w^3+6(x-y)^3$" ijmtt, Vol 20,April 2015,pg 75-84.