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Anti Gaussian Quadrature For Real Definite Integral

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Abstract- A higher degree precision quadrature rule has been constructed with the problem of determining the approximated solution of real definite integrals using anti Lobatto five point rule and anti Clenshaw Curtis seven point rule which has been further compared with another mixed quadrature rule for different integrals with their anti Gaussian rules. Some numerical examples are provided to illustrate the accuracy and comparison of absolute error of proposed rule with constituent quadrature rules.

Index Terms- Numerical Integration, Anti-Gaussian quadrature rules, Gaussian quadrature rules, Mixed quadrature rule, Degree of precision

1. INTRODUCTION

Numerical integration is the approximate numerical computation of an integral. Gauss quadrature could be a wide spread approach to approximate the value of an integral determined by a measure with support on the real axis.

Dirk P. Laurie [1] was first coined the idea of anti-Gaussian quadrature formula. The $(n+1)$ point formula of anti-Gaussian quadrature rule of degree $(2n-1)$ integrates the polynomials of degree up to $(2n+1)$ with an error equal in magnitude but of opposite sign to that of the Gaussian n point formula. It meant the application is to evaluate the error occurred in Gaussian integration by having the distinction between the results occurred from the two formulas. The anti Gaussian formula has positive weights and the nodes within the integration interval and reticulate by the corresponding Gaussian formula.

The method of mixing quadrature rule is based on forming a higher degree precision quadrature rule by taking the convex combination of two lower precision quadrature rules. The concept of mixed quadrature was first introduced by Das and Pradhan [5]. Various research work have been done in this area towards the numerical evaluation of real definite integrals. Among them, Jena and Nayak [6] has applied mixed quadrature rule to find the approximate solution of non linear Fredholm integral equation of second kind, Jena and Dash [7] has established mixed quadrature over sphere with finite element approach. Dash and Das [8],[11] has proposed identification of some Clenshaw- Curtis rule with Fejer rules and also in adaptive field. Tripathy etal. [9] used a mixed quadrature of Lobatto four point rule with Gaussian quadrature for approximate evaluation of real definite integrals. Singh and Dash [2] has formed a mixed quadrature rule using an anti Lobatto four point rule. The proposed work is a comparison with [2] . They have used only the ant Gaussian with Gaussian rule for the mixed quadrature where as we have applied Gaussian as well as anti Gaussian rule for the mixed quadrature rule. The Lobatto four point rule and Clenshaw Curtis five point Gaussian rule have chosen for derivation of Lobatto five point rule and Clenshaw Curtis seven point anti Gaussian rule each of degree of precision five respectively.

$$G_w^{(n)} = \sum_{j=1}^{n} w_j^{(n)} f\left(x_j^{(n)}\right) \tag{1.1}$$

of degree $(2n-1)$ for the integral

$$I = \int_a^b f(x)w(x)dx \tag{1.2}$$

$G_w^{(n)}(p) = I(p),\ \forall p \in P^{2n-1}$ where P^m is the space of polynomial of degree not greater than m. If $H^{(n+1)} = \sum_{j=1}^{n+1} \lambda_j f(\zeta_j)$ is an anti Gaussian formula for $(n+1)$ point and $G^{(n)}(p)$ be n point Gaussian formula, then by hypothesis $I(p) - H^{(n+1)}(p) = -(I(p) - G^{(n)}(p))$ where p defined as polynomial of degree $\le 2n+1$.

The organization of the paper is as follows. In section 2, construction of anti Gaussian Lobatto five point has been described. Section 3 contains construction of anti Gaussian Clenshaw Curtis seven point rule and the mixed quadrature rule has been formed for different constituent rules in section 4. Numerical results are verified in section 5. Section 6 has drawn some conclusion.

2. CONSTRUCTION OF ANTI-LOBATTO FIVE POINT RULE FROM LOBATTO FOUR POINT RULE

We choose the Lobatto four point rule,

$$L_w{}^4(f)=\frac{1}{6}\left[\begin{array}{l}\{f(-1)+f(1)\}+\\ 5\left\{f\left(-\frac{1}{\sqrt{5}}\right)+f\left(\frac{1}{\sqrt{5}}\right)\right\}\end{array}\right] \qquad (2.1)$$

to develop a five point Lobatto rule $RH_w{}^5(f)$ from four point Lobatto rule $L_w{}^4(f)$. Using the principle $I(p)-H^{(n+1)}(p)=-\left(I(p)-G^{(n)}(p)\right)$ as adopted in Dirk. P. Laurie [1], we obtain

$$RH_w{}^5(f)=2\int_{-1}^{1} f(x)dx-L_w{}^4(f) \qquad (2.2)$$

$$\begin{array}{l}\alpha_1 f(-1)+\alpha_2 f(\xi_1)+\alpha_3 f(\xi_2)+\alpha_4 f(\xi_3)+\\ \alpha_5 f(1)=2\int_{-1}^{1} f(x)dx-L_w{}^4(f)\end{array} \qquad (2.3)$$

where

$$\begin{array}{l}RH_w{}^5(f)=\alpha_1 f(-1)+\alpha_2 f(\xi_1)+\alpha_3 f(\xi_2)\\ +\alpha_4 f(\xi_3)+\alpha_5 f(1)\end{array}$$

In order to obtain the unknown weights and nodes, a rule of precision five has been considered. Since the rule has been integrated for polynomial of degree seven, we obtain following system of eight equations having eight unknowns namely,

$\alpha_j\,(j=1,2,3,4,5)$, $\xi_j\,(j=1,2,3)$ for $f(x)=x^j\,(j=0,1,2,3,4,5,6,7)$

The system of equations are

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=2 \qquad (2.4)$$

$$-\alpha_1+\alpha_2\xi_1+\alpha_3\xi_2+\alpha_4\xi_3+\alpha_5=0 \qquad (2.5)$$

$$-\alpha_1+\alpha_2\xi_1{}^2+\alpha_3\xi_2{}^2+\alpha_4\xi_3{}^2+\alpha_5=\frac{2}{3} \qquad (2.6)$$

$$-\alpha_1+\alpha_2\xi_1{}^3+\alpha_3\xi_2{}^3+\alpha_4\xi_3{}^3+\alpha_5=0 \qquad (2.7)$$

$$-\alpha_1+\alpha_2\xi_1{}^4+\alpha_3\xi_2{}^4+\alpha_4\xi_3{}^4+\alpha_5=\frac{2}{5} \qquad (2.8)$$

$$-\alpha_1+\alpha_2\xi_1{}^5+\alpha_3\xi_2{}^5+\alpha_4\xi_3{}^5+\alpha_5=0 \qquad (2.9)$$

$$-\alpha_1+\alpha_2\xi_1{}^6+\alpha_3\xi_2{}^6+\alpha_4\xi_3{}^6+\alpha_5=\frac{118}{525}$$

(2.10)

$$-\alpha_1+\alpha_2\xi_1{}^7+\alpha_3\xi_2{}^7+\alpha_4\xi_3{}^7+\alpha_5=0 \qquad (2.11)$$

The solution of above system of equations are

$$\alpha_1=\alpha_5=-\frac{1}{18},\ \alpha_2=\alpha_3=\frac{245}{414},$$

$$\alpha_4=\frac{64}{69},\ \xi_1=\sqrt{\frac{23}{35}},\quad \xi_2=-\sqrt{\frac{23}{35}},\quad \xi_3=0$$

Hence the anti Lobatto five rule becomes

$$RH_w{}^5(f)=\left[\begin{array}{l}\frac{245}{414}\left\{\begin{array}{l}f\left(-\sqrt{\frac{23}{35}}\right)\\ +f\left(\sqrt{\frac{23}{35}}\right)\end{array}\right\}\\ +\frac{64}{69}f(0)\\ -\frac{1}{18}\left\{\begin{array}{l}f(-1)\\ +f(1)\end{array}\right\}\end{array}\right] \qquad (2.12)$$

The error associated with the rule is computed as

$$\begin{array}{l}EH_w{}^5(f)=\int_{-1}^{1} f(x)dx-RH_w{}^5(f)\\ =\frac{32}{525\times 6!}f^{vi}(0)+\frac{6208}{55125\times 8!}f^{viii}(0)\\ +\frac{348714}{2358125\times 10!}f^{x}(0)+\ldots.\end{array}$$

3. CONSTRUCTION OF ANTI-CLENSHAW CURTIS SEVEN POINT RULE FROM CLENSHAW CURTIS FIVE POINT RULE

We choose the Clenshaw-Curtis five point rule,

$$C_w{}^5(f)=\frac{1}{15}\left[\begin{array}{l}\{f(-1)+f(1)\}+\\ 8\left\{\begin{array}{l}f\left(-\frac{1}{\sqrt{2}}\right)\\ +f\left(\frac{1}{\sqrt{2}}\right)\end{array}\right\}+12f(0)\end{array}\right] \qquad (3.1)$$

and develop a seven point Clenshaw-Curtis rule $RH_w{}^7(f)$ from five point Clenshaw-Curtis rule $C_w{}^5(f)$

Using the principle $I(p)-H^{(n+1)}(p)=-\left(I(p)-G^{(n)}(p)\right)$ as adopted in Dirk. P. Laurie [1], we obtain

$$RH_w{}^7(f)=2\int_{-1}^{1} f(x)dx-C_w{}^5(f) \qquad (3.2)$$

$$\alpha_1 f(-1)+\alpha_2 f(\xi_1)+\alpha_3 f(\xi_2)+\alpha_4 f(\xi_3)+\alpha_5 f(1)=2\int_{-1}^{1} f(x)dx-C_w^{5}(f) \quad (3.3)$$

where $RH_w^{7}(f)=\alpha_1 f(-1)+\alpha_2 f(\xi_1)+\alpha_3 f(\xi_2)+\alpha_4 f(\xi_3)+\alpha_5 f(1)$

In order to obtain the unknown weights and nodes, a rule of precision five has been considered. Since the rule has been integrated for polynomial of degree seven, we obtain following system of eight equations having eight unknowns namely, $\alpha_j\ (j=1,2,3,4,5)$, $\xi_j\ (j=1,2,3)$ for $f(x)=x^j\ (j=0,1,2,3,4,5,6,7)$

The system of equations are

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=2 \quad (3.4)$$

$$-\alpha_1+\alpha_2\xi_1+\alpha_3\xi_2+\alpha_4\xi_3+\alpha_5=0 \quad (3.5)$$

$$-\alpha_1+\alpha_2\xi_1^{2}+\alpha_3\xi_2^{2}+\alpha_4\xi_3^{2}+\alpha_5=\frac{2}{3} \quad (3.6)$$

$$-\alpha_1+\alpha_2\xi_1^{3}+\alpha_3\xi_2^{3}+\alpha_4\xi_3^{3}+\alpha_5=0 \quad (3.7)$$

$$-\alpha_1+\alpha_2\xi_1^{4}+\alpha_3\xi_2^{4}+\alpha_4\xi_3^{4}+\alpha_5=\frac{2}{5} \quad (3.8)$$

$$-\alpha_1+\alpha_2\xi_1^{5}+\alpha_3\xi_2^{5}+\alpha_4\xi_3^{5}+\alpha_5=0 \quad (3.9)$$

$$-\alpha_1+\alpha_2\xi_1^{6}+\alpha_3\xi_2^{6}+\alpha_4\xi_3^{6}+\alpha_5=\frac{32}{105} \quad (3.10)$$

$$-\alpha_1+\alpha_2\xi_1^{7}+\alpha_3\xi_2^{7}+\alpha_4\xi_3^{7}+\alpha_5=0 \quad (3.11)$$

The solution of above system of equations are

$$\alpha_1=\alpha_5=\frac{17}{135},\ \alpha_2=\alpha_3=\frac{392}{675},$$

$$\alpha_4=\frac{44}{75},\ \xi_1=\sqrt{\frac{5}{14}},\quad \xi_2=-\sqrt{\frac{5}{14}},\quad \xi_3=0$$

Hence the anti Clenshaw-Curtis seven point rule becomes

$$RH_w^{7}(f)=\left[\begin{array}{l}\frac{392}{675}\left\{\begin{array}{l}f\left(-\sqrt{\frac{5}{14}}\right)\\+f\left(\sqrt{\frac{5}{14}}\right)\end{array}\right\}\\+\frac{44}{75}f(0)\\+\frac{17}{135}\{f(-1)+f(1)\}\end{array}\right] \quad (3.12)$$

The error associated with the rule is computed as

$$EH_w^{7}(f)=\int_{-1}^{1} f(x)dx-RH_w^{7}(f)$$

$$=-\frac{2}{105\times 6!}f^{vi}(0)-\frac{107}{2205\times 8!}f^{viii}(0)-\frac{2897}{37730\times 10!}f^{x}(0)\ldots$$

4. CONSTRUCTION OF MIXED QUADRATURE RULE

In this section we have constructed a mixed quadrature rule taking four constituent rules and error analysis has been made.

4.1 Anti-Clenshaw Curtis seven point rule with anti Lobatto five point rule

We have anti-Clenshaw Curtis seven point rule

$$RH_w^{7}(f)=\left[\begin{array}{l}\frac{392}{675}\left\{f\left(-\sqrt{\frac{5}{14}}\right)+f\left(\sqrt{\frac{5}{14}}\right)\right\}\\+\frac{44}{75}f(0)+\frac{17}{135}\{f(-1)+f(1)\}\end{array}\right]$$

and the anti Lobatto five rule

$$RH_w^{5}(f)=\left[\begin{array}{l}\frac{245}{414}\left\{f\left(-\sqrt{\frac{23}{35}}\right)+f\left(\sqrt{\frac{23}{35}}\right)\right\}\\+\frac{64}{69}f(0)-\frac{1}{18}\{f(-1)+f(1)\}\end{array}\right]$$

where $RH_w^{5}(f)$ and $RH_w^{7}(f)$ is of degree of precision five and $EH_w^{5}(f)$ and $EH_w^{7}(f)$ denote the corresponding errors by the rules $RH_w^{5}(f)$ and $RH_w^{7}(f)$ for the integrals $I(f)$ respectively. Now

$$I=RH_w^{5}+EH_w^{5} \quad (4.1.1)$$

$$I=RH_w^{7}+EH_w^{7} \quad (4.1.2)$$

By Maclaurin's expansion of function in equation (4.1) and (4.2), we have

$$EH_w^{\,5}(f)=\frac{32}{525\times 6!}f^{vi}(0)+\frac{6208}{55125\times 8!}f^{viii}(0)+\frac{348714}{2358125\times 10!}f^{x}(0)+\ldots \quad (4.1.3)$$

$$EH_w^{\,7}(f)=-\frac{2}{105\times 6!}f^{vi}(0)-\frac{107}{2205\times 8!}f^{viii}(0)-\frac{2897}{37730\times 10!}f^{x}(0)\ldots \quad (4.1.4)$$

Eliminating $f^{vi}(0)$ from equation $(4.1.3)$ and $(4.1.4)$ we have

$$I=\frac{1}{21}\left[5RH_w^{\,5}(f)+16RH_w^{\,7}(f)\right]+\frac{1}{21}\left[5EH_w^{\,5}(f)+16EH_w^{\,7}(f)\right]$$

$$I(f)=RH_w^{\,5}H_w^{\,7}(f)+EH_w^{\,5}H_w^{\,7}(f) \quad (4.1.5)$$

$$RH_w^{\,5}H_w^{\,7}(f)=\frac{1}{21}\begin{bmatrix}5RH_w^{\,5}(f)\\+16RH_w^{\,7}(f)\end{bmatrix} \quad (4.1.6)$$

which is the estimated mixed rule of precision seven. The truncated error for the approximation is

$$EH_w^{\,5}H_w^{\,7}(f)=\frac{1}{21}\begin{bmatrix}5EH_w^{\,5}(f)\\+16EH_w^{\,7}(f)\end{bmatrix} \quad (4.1.7)$$

$$EH_w^{\,5}H_w^{\,7}(f)=-\frac{112}{11025\times 8!}f^{viii}(0)-\frac{230686}{9904125\times 10!}f^{x}(0)\ldots$$

4.2 Anti Lobatto five point rule with Fejer's five point second rule

We have anti Lobatto five point

$$RH_w^{\,5}(f)=\begin{bmatrix}\frac{245}{414}\left\{\begin{matrix}f\left(-\sqrt{\frac{23}{35}}\right)\\+f\left(\sqrt{\frac{23}{35}}\right)\end{matrix}\right\}\\+\frac{64}{69}f(0)\\-\frac{1}{18}\{f(-1)+f(1)\}\end{bmatrix} \quad (4.2.1)$$

and Fejer's five point second rule

$$R_{2f5}(f)=\frac{2}{45}\begin{bmatrix}7\left\{f\left(-\frac{\sqrt{3}}{2}\right)+f\left(\frac{\sqrt{3}}{2}\right)\right\}\\+13f(0)\\+9\left\{f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right\}\end{bmatrix} \quad (4.2.2)$$

where the rules $RH_w^{\,5}(f)$ and $R_{2f5}(f)$ is of precision five and $EH_w^{\,5}(f)$ and $E_{2f5}(f)$ is the errors for the integrals $I(f)$ due to the rules $RH_w^{\,5}(f)$ and $R_{2f5}(f)$ respectively. Now

$$I=RH_w^{\,5}+EH_w^{\,5}(f) \quad (4.2.1)$$

$$I(f)=R_{2f5}(f)+E_{2f5}(f) \quad (4.2.2)$$

Expanding equation $(4.2.1)$ and $(4.2.2)$ by Maclaurin's expansion, we have

$$EH_w^{\,5}(f)=\frac{32}{525\times 6!}f^{vi}(0)+\frac{6208}{55125\times 8!}f^{viii}(0)+\frac{348714}{2358125\times 10!}f^{x}(0)+\ldots \quad (4.2.3)$$

$$E_{2f5}(f)=\frac{3}{280\times 6!}f^{vi}(0)+\frac{1}{45\times 8!}f^{viii}(0)+\frac{47}{1408\times 10!}f^{x}(0)\ldots \quad (4.2.4)$$

Eliminating $f^{vi}(0)$ from equation $(4.2.3)$ and $(4.2.4)$ we have

$$I(f)=\frac{1}{1477}\left[1792\,R_{2f5}(f)-315\,RH_w^{\ 5}(f)\right]+\frac{1}{1477}\left[1792\,E_{2f5}(f)-315\,EH_w^{\ 5}(f)\right]$$

$$I(f)=RH_w^{\ 5}R_{2f5}(f)+E\ H_w^{\ 5}R_{2f5}(f) \qquad (4.2.5)$$

$$RH_w^{\ 5}R_{2f5}(f)(f)=\frac{1}{1477}\begin{bmatrix}1792\,R_{2f5}(f)- \\ 315\,RH_w^{\ 5}(f)\end{bmatrix} \qquad (4.2.6)$$

which is the mixed rule of precision seven and the truncated error for this approximation is

$$EH_w^{\ 5}R_{2f5}(f)=\frac{1}{1477}\begin{bmatrix}1792\,E_{2f5}(f) \\ -315\,EH_w^{\ 5}(f)\end{bmatrix} \qquad (4.2.7)$$

$$EH_w^{\ 5}R_{2f5}(f)=\frac{6848}{2326275\times 8!}f^{viii}(0)+\frac{2740}{207\times 10!}f^{x}(0)\ldots$$

4.3 Anti-Clenshaw Curtis seven point rule with Fejer's five point second rule

We have anti Clenshaw-Curtis seven point rule

$$RH_w^{\ 7}(f)=\begin{bmatrix}\frac{392}{675}\left\{f\left(-\sqrt{\frac{5}{14}}\right)+f\left(\sqrt{\frac{5}{14}}\right)\right\} \\ +\frac{44}{75}f(0)+\frac{17}{135}\{f(-1)+f(1)\}\end{bmatrix}$$

and Fejer's five point second rule

$$R_{2f5}(f)=\frac{2}{45}\begin{bmatrix}7\left\{f\left(-\frac{\sqrt{3}}{2}\right)+f\left(\frac{\sqrt{3}}{2}\right)\right\} \\ +13f(0)+9\left\{f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right\}\end{bmatrix}$$

Where $RH_w^{\ 7}(f)$ and $R_{2f5}(f)$ is of precision five and $EH_w^{\ 7}(f)$ and $E_{2f5}(f)$ is the error for the integrals $I(f)$ for the rules $RH_w^{\ 7}(f)$ and $R_{2f5}(f)$ respectively. Now

$$I=RH_w^{\ 7}+EH_w^{\ 7} \qquad (4.3.1)$$

$$I=R_{2f5}+E_{2f5} \qquad (4.3.2)$$

Expanding equation (4.1) and (4.2) by Maclaurin's expansion , we have

$$EH_w^{\ 7}(f)=-\frac{2}{105\times 6!}f^{vi}(0)-\frac{107}{2205\times 8!}f^{viii}(0)-\frac{2897}{37730\times 10!}f^{x}(0)\ldots \qquad (4.3.3)$$

$$E_{2f5}(f)=\frac{3}{280\times 6!}f^{vi}(0)+\frac{1}{45\times 8!}f^{viii}(0)+\frac{47}{1408\times 10!}f^{x}(0)\ldots \qquad (4.3.4)$$

Eliminating $f^{vi}(0)$ from equation $(4.3.3)$ and $(4.3.4)$, we have

$$I=\frac{1}{175}\left[112\,R_{2f5}(f)+63\,RH_w^{\ 7}(f)\right]+\frac{1}{175}\left[112\,E_{2f5}(f)+63\,EH_w^{\ 7}(f)\right]$$

$$I(f)=RH_w^{\ 7}R_{2f5}(f)+E\ H_w^{\ 7}R_{2f5}(f) \qquad (4.3.5)$$

$$RH_w^{\ 7}R_{2f5}(f)=\frac{1}{175}\left[112\ R_{2f5}(f)+63RH_w^{\ 7}(f)\right] \qquad (4.3.6)$$

which is the mixed rule of precision seven and the truncated error generated for the approximation is

$$EH_w^{\ 7}R_{2f5}(f)=\frac{1}{175}\begin{bmatrix}112\,E_{2f5}(f) \\ +63EH_w^{\ 7}(f)\end{bmatrix} \qquad (4.3.7)$$

$$EH_w^{\ 7}R_{2f5}(f)=-\frac{179}{55125\times 8!}f^{viii}(0)-\frac{260557}{41503000\times 10!}f^{x}(0)\ldots$$

4.4 Error Analysis and error bounds of mixed quadrature rules

Theorem-1

Let the smooth function $f(x)$ is defined on $-1\le x\le 1$, then the error $EH_w^{\ 5}H_w^{\ 7}(f)$ due to the mixed quadrature rule $RH_w^{\ 5}H_w^{\ 7}(f)$ is given by

$$EH_w{}^5H_w{}^7(f)=-\frac{112}{11025\times 8!}f^{viii}(0)$$
$$-\frac{230686}{9904125\times 10!}f^{x}(0)....$$

Proof:

The proof of theorem follows from equation $(4.1.7)$.

Theorem-2

Let the smooth function $f(x)$ is defined on $-1\le x\le 1$, then the error $EH_w{}^5R_{2f5}(f)$ due to the mixed quadrature rule $RH_w{}^5R_{2f5}(f)$ is given by

$$EH_w{}^5R_{2f5}(f)=\frac{6848}{2326275\times 8!}f^{viii}(0)$$
$$+\frac{2740}{207\times 10!}f^{x}(0)....$$

Proof:

The proof of theorem follows from equation $(4.2.7)$.

Theorem-3

Let the smooth function $f(x)$ is defined on $-1\le x\le 1$, then the error $EH_w{}^7R_{2f5}(f)$ due to the mixed quadrature rule $RH_w{}^7R_{2f5}(f)$ is given by

$$EH_w{}^7R_{2f5}(f)=-\frac{179}{55125\times 8!}f^{viii}(0)$$
$$-\frac{260557}{41503000\times 10!}f^{x}(0)....$$

Proof:

The proof of Theorem follows from equation $(4.3.6)$.

Theorem-4

The truncated error bound for $EH_w{}^5H_w{}^7(f)=I(f)-RH_w{}^5H_w{}^7(f)$ is evaluated by $\left|EH_w{}^5H_w{}^7(f)\right|\le\frac{64M}{105\times 6!}$ where $M=\max_{-1\le x\le 1}\left|f^{vii}(x)\right|$.

Proof

$$EH_w{}^5(f)=\frac{32}{525\times 6!}f^{vi}(\eta_1),\qquad \eta_1\in[-1,1]$$

$$EH_w{}^7(f)=-\left(\frac{2}{105\times 6!}\right)f^{vi}(\eta_2),\qquad \eta_2\in[-1,1]$$

$$EH_w{}^5H_w{}^7(f)=\frac{32}{105\times 6!}\left[f^{vi}(\eta_2)-f^{vi}(\eta_1)\right]$$

(Conte & Boor $[10]$)

$$\left|EH_w{}^5H_w{}^7(f)\right|\cong\frac{32}{105\times 6!}\left[f^{vi}(d)-f^{vi}(c)\right]$$
$$=\frac{32}{105\times 6!}\int_{-1}^{1}f^{vii}(x)dx$$
$$=\frac{32}{105\times 6!}(d-c)f^{vii}(\gamma)$$

for

some $\gamma\in[-1,1]$

where $|d-c|\le 2$

$$\left|EH_w{}^5H_w{}^7(f)\right|\le\frac{64}{105\times 6!}f^{vii}(\gamma)$$

Hence $\left|EH_w{}^5H_w{}^7(f)\right|\le\frac{64M}{105\times 6!}$ where $M=\max_{-1\le x\le 1}\left|f^{vii}(x)\right|$.

Theorem-5

The truncated error bound for $EH_w{}^5R_{2f5}(f)=I(f)-RH_w{}^5R_{2f5}(f)$ is evaluated by $\left|EH_w{}^5R_{2f5}(f)\right|\le\frac{1344M}{51695\times 6!}$ where $M=\max_{-1\le x\le 1}\left|f^{vii}(x)\right|$.

Proof:

$$EH_w{}^5(f)=\frac{32}{525\times 6!}f^{vi}(\eta_1),\qquad \eta_1\in[-1,1]$$

$$ER_{2f5}(f)=\frac{3}{280\times 6!}f^{vi}(\eta_2),\qquad \eta_2\in[-1,1]$$

$$EH_w{}^5R_{2f5}(f)=\frac{672}{51695\times 6!}\left[f^{vi}(\eta_2)-f^{vi}(\eta_1)\right]$$

As per Theorem-4,

$$\left|EH_w{}^5R_{2f5}(f)\right|\cong\frac{672}{51695\times 6!}\left[f^{vi}(d)-f^{vi}(c)\right]$$
$$=\frac{672}{51695\times 6!}\int_{-1}^{1}f^{vii}(x)dx$$
$$=\frac{672}{51695\times 6!}(d-c)f^{vii}(\gamma)$$

for some $\gamma\in[-1,1]$

where $|d-c| \le 2$

$$\left|EH_w{}^5 R_{2f5}(f)\right| \le \frac{1344}{51695 \times 6!} f^{vii}(\gamma)$$

Hence $\left|EH_w{}^5 R_{2f5}(f)\right| \le \frac{1344M}{51695 \times 6!}$

where $M = \max_{-1 \le x \le 1} \left| f^{vii}(x) \right|$.

Theorem-6

The truncated error bound for $EH_w{}^7 R_{2f5}(f) = I(f) - RH_w{}^7 R_{2f5}(f)$ is evaluated by $\left|EH_w{}^7 R_{2f5}(f)\right| \le \frac{84M}{6125 \times 6!}$

where $M = \max_{-1 \le x \le 1} \left| f^{vii}(x) \right|$

Proof

$$EH_w{}^7(f) = -\frac{2}{105 \times 6!} f^{vi}(\eta_1), \qquad \eta_1 \in [-1,1]$$

$$ER_{2f5}(f) = \frac{3}{280 \times 6!} f^{vi}(\eta_2), \qquad \eta_2 \in [-1,1]$$

$$EH_w{}^7 R_{2f5}(f) = \frac{42}{6125 \times 6!}\left[f^{vi}(\eta_2) - f^{vi}(\eta_1)\right]$$

As per Theorem-4 and Theorem-5,

$$\left|EH_w{}^7 R_{2f5}(f)\right| \cong \frac{42}{6125 \times 6!}\left[f^{vi}(d) - f^{vi}(c)\right]$$

$$= \frac{42}{6125 \times 6!} \int_{-1}^{1} f^{vii}(x)dx = \frac{42}{6125 \times 6!} (d-c) f^{vii}(\gamma)$$

$for some\ \gamma \in [-1, 1]$

where $|d-c| \le 2$

$$\left|EH_w{}^7 R_{2f5}(f)\right| \le \frac{84}{6125 \times 6!} f^{vii}(\gamma)$$

Hence $\left|EH_w{}^7 R_{2f5}(f)\right| \le \frac{84M}{6125 \times 6!}$

where $M = \max_{-1 \le x \le 1} \left| f^{vii}(x) \right|$.

5. Numerical Results

In this section some numerical examples are taken to validate our proposed work. The absolute error shows a solid comparison between different mixed quadrature rules $\left(R_{Hw^5 Hw^7}(f)\right)$, $\left(R_{Hw^5 2f5}(f)\right)$ and $\left(R_{Hw^5 2f5}(f)\right)$ which provides better approximation to exact results than the constituent rules for different integrals.

The approximate value of the following integrals have been calculated which has given in Table-1.

$I_1 = \int_{-1}^{1} e^x\,dx$ =2.350402387287603, $I_2 = \int_{0}^{1} e^{x^2}\,dx$ =1 .462651745907181 ,

$I_3 = \int_{0}^{1} e^{-x^2}\,dx$ =0.746824132812427,

$I_4 = \int_{1}^{3} \frac{\sin^2 x}{x}\,dx$ =0.794825180668111,

$I_5 = \int_{0}^{1} \sqrt{x}\,dx$ =0.666666666666667,

$I_6 = \int_{0}^{1} \sqrt{x}\sin x\,dx$ =0.364221932032132,

$I_7 = \int_{0}^{1} \sin\sqrt{\pi x}\,dx$ =0.849726325420498,

$I_8 = \int_{0}^{1} x^{10}\cos x^{16}\,dx$ =0.049121729517639

$I_9 = \int_{2}^{3} \frac{\ln\sqrt{x}}{x}\,dx$ =0.181623986723595,

$I_{10} = \int_{1}^{2} \frac{\cosh x}{\sqrt{x}}\,dx$ =1.980270250563978,

$I_{11} = \int_{3}^{4} \cosh^{-1} x \log x\,dx$ =2.40709642933493

Table-1(Comparison of absolute error for anti mixed quadrature rule $\left(R_{Hw^5 Hw^7}(f)\right)$ with mixed rule of Gaussian and anti Gaussian $\left(R_{Hw^5 2f5}(f)\right)$ and $\left(R_{Hw^5 2f5}(f)\right)$).

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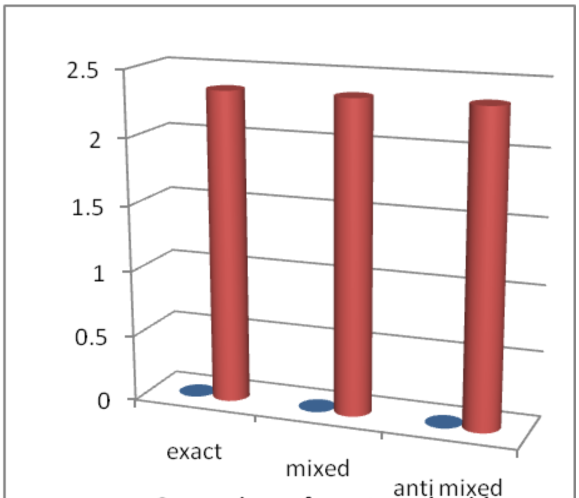
I	$E_{Hw^5}(f)$	$E_{2f5}(f)$	$E_{Hw^7}(f)$	$E_{Hw^5 Hw^7}(f)$	$E_{Hw^5 2f5}(f)$	$E_{Hw^7 2f5}(f)$
I_1	0.00 0087 4902 6950 3	0.000 01544 13877 45	0.000 02767 99203 75	0.000 00025 84465 95	0.0000 00075 51248 8	0.00 0000 0822 8317 9
I_2	0.00 0326 3044	0.000 05844 11378	0.000 10765 89892	0.000 00433 43685	0.0000 01313 89805	0.00 0001 3549

	17884	70	45	00	6	07891
I_3	0.000012495653673	0.000002130577328	0.000003521672235	0.000000291976791	0.000000079983978	0.000000095767485
I_4	0.000227765036674	0.000039397092307	0.000067544735297	0.000002767115171	0.000000776165970	0.000000898034370
I_5	0.006073267446686	0.001329575316137	0.006520005019592	0.003521606813335	0.000317887420996	0.00149627360425
I_6	0.000304580602496	0.000070921838863	0.000182326655472	0.00006639635596	0.000021089401121	0.000020247619098
I_7	0.011050152657705	0.002427915268385	0.011730850272801	0.006304570070298	0.000583410486776	0.002671160326442
I_8	0.005114390495036	0.001419563021141	0.007120856162705	0.006643126241831	0.002813060216534	0.005836718135397
I_9	0.000000653438812	0.000000116145968	0.000000211023953	0.000000005199485	0.000000001557447	0.000000001635203
I_{10}	0.0000101431798	0.000001824260313	0.000003380641378	0.000000160683946	0.000000050083164	0.000000049504296
I_{11}	0.000000069152765	0.000000012230125	0.000000022011393	0.000000000305641	0.000000000090225	0.000000000968222



Comparison of exact result with approximate result for I1

(Fig.-1)

6. CONCLUSION

The efficiency of our proposed rule is a good agreement with the exact result, which has been drawn from Table-1 numerically as well as from FIG-1 graphically. Error analysis of these methods besides the test numerical examples provide a solid foundation to compare between anti Gaussian-anti Gaussian mixed quadrature rule and mixed Gaussian-anti Gaussian quadrature rule for numerical estimation of real definite integrals. The main advantages of the presented method is its simple computational evaluations which is wholly competitive in comparison with the Gaussian methods.

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