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Numerical solution of Volterra integro-differential equations of fractional order using Mahgoub Adomian Decomposition Method

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Abstract-In this article, MahgoubAdomian decomposition method has been developed to find the numerical solution of nonlinear Volterraintegro differential equations of fractional order. The proposed method is the combined form of Mahgoub transform and Adomian decomposition method. The nonlinear term has been handled with the help of Adomian polynomials. Here, the fractional derivatives are described in Caputo sense. The results of illustrative examples reveal that our proposed method is good, reliable and effective.

Index Terms-Caputo derivative;Mahgoub Transform; Fractional volterraintegro-differential Equations; Adomian polynomials.

1. INTRODUCTION

Fractional differential equations are generalized from classical integer order which are obtained by replacing the integer order derivatives by fractional. Fractional integro differential equations have found extensive application in various fields of science and engineering such as image processing, viscoelastic, thermal systems, fluid flow and mechanics [3][5][7]. Many researchers have shown their interest in finding the numerical solution of linear and nonlinear fractional integro differential equations. Most of the nonlinear Fractional integro-differential equations do not have exact solution, so some numerical and approximation technique must be used.Variational iteration method [6], Fractional Differential Transform Method [1], Collocation Method [4][8] and wavelet method [10][12] are the some of the methods to solve fractional integro differential equations.

The Adomiandecompoisition method (ADM) is one of the powerful methods to find the approximate solution of linear and nonlinear Fractional integro differential equations. Wang et. al [9] applied the Adomian polynomial to solve the nonlinear Volterraintegro differential equation of fractional order. Yang and Hou [11] delivered the numerical solution of Fractional integro differential equations using Laplace decomposition method.

In this paper MahgoubAdomian Decomposition method (MADM) have been proposed to determine the numerical solution of nonlinear Volterraintegro-differential equation of fractional order.

This paper has been organized as follows. Section 2 introduces some fundamental definitions and properties of the fractional calculus and Mahgoub transform. Section 3 constructs our method to approximate the solution of the Volterraintegro-differential equations of fractional order. Section 4 presents some numerical examples to illustrate the accuracy of our method.

2. PRELIMINARIES AND NOTATIONS

In this section, fundamental definitions and properties of fractional calculus and Mahgoub transform have been introduced.

2.1. *Definition*

A real function $f(t), t > 0$ is said to be in the space $\mathbb{C}_{\mu},\ \mu \epsilon \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$ where $f_1(t) \epsilon \mathbb{C}[0,\infty)$ and it is said to be in the space $\mathbb{C}_{\mu}^{n}$ if and only if $f^{(n)} \epsilon \mathbb{C}_{\mu},\ n \epsilon \mathbb{N}$.

2.2. *Definition*

The Riemann Liouville fractional integral $I^{\alpha} f(t)$ of order $\alpha \in R, \alpha > 0$ of function$f(t) \in C_{\mu}, \mu \geq -1$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad t > 0 \quad (1)$$

$$I^0 f(t) = f(t)$$

2.3. *Definition*

The Caputo fractional derivative of a function $f(t)$of order $\alpha \in R, \alpha > 0$ is given by

$${}^{c}D^{\alpha} \quad f(t) = I^{n-\alpha} D^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau)d\tau, t > 0 \ (2)$$

where $n-1 \leq \alpha \leq n, n \epsilon \mathbb{N}^{+}$ and $\Gamma(.)$ denotes the Gamma function.

2.4. *Definition*

Mahgoub transform is defined on the set of continuous functions and exponential order. We consider functions in the set A defined by

$$A=\left\{\begin{matrix} f(t): |f(t)| < Pe^{\frac{|t|}{\epsilon_i}} if\ t \in (-1)^i \times [0,\infty), \\ i=1,2;\ \epsilon_i > 0 \end{matrix}\right\}$$

(3)

where ϵ_1, ϵ_2 may be finite or infinite and the constant P must be finite.

Let $f \in A$, then Mahgoub transform is defined as

$M[f(t)] = H(u) = u\int_0^\infty f(t)e^{-ut}dt$(4)

for $t \geq 0, \epsilon_1 \leq u \leq \epsilon_2$

2.5. Theorem

Let $F(u)$ and $G(u)$ denote the Mahgoub transform of $f(t)$ and $g(t)$ respectively. If

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)d\tau$$

where $*$ denotes convolution of f and g, then the Mahgoub transform of the convolution of $f(t)$ and $g(t)$ is

$M[f(t)*g(t);u] = \frac{1}{u}F(u)G(u)$ (5)

2.6. Theorem[2]

Let $n\epsilon\mathbb{N}$ and $\alpha > 0$ be such that $n-1<\alpha\leq n$ and $H(u)$ be the Mahgoub transform of the function $f(t)$, then the Mahgoub transform of Caputo fractional derivative of $f(t)$ of order α is given by

$M[{}_0^cD_t^\alpha f(t)] = u^\alpha H(u) - \sum_{k=0}^{n-1} u^{\alpha-k} f^{(k)}(0)$(6)

3. ANALYSIS OF THE METHOD

Consider the Volterraintegro-differential equation of fractional order

${}^cD^\alpha\ \ y(t)p(t)y(t) = g(t) + \lambda\int_0^t k(t,x)F(y(x))dx$

(7)

for $t\ \epsilon\ [0,1]$, with the initial conditions $y^{(i)}(0)=\delta_i$, $i=0,1,\dots,n-1, n-1<\alpha\leq n,\ n\ \epsilon\ \mathbb{N}$.

where $g\ \epsilon\ L^2([0,1]), p\ \epsilon\ L^2([0,1]), k\epsilon\ L^2([0,1]^2)$ are known functions, $y(t)$ is the unknown function, $F(y)$ is the nonlinear function and D^α is the Caputo fractional differential operator of order α.

Applying Mahgoub transform in Eqn. (7)

$$M\left[{}^cD^\alpha\ \ y(t)\right] = M[p(t)y(t)] + M[g(t)] + M\left[\lambda\int_0^t k(t,x)F(y(x))dx\right]$$

Using the property of Mahgoub transform of Caputo fractional derivative we get

$$M[y(t)] = \sum_{k=0}^{n-1} u^{-k}y^k(0) + \frac{1}{u^\alpha}M[p(t)y(t)] + \frac{1}{u^\alpha}M[g(t)]$$

$+\frac{1}{u^\alpha}M\left[\lambda\int_0^t k(t,x)F(y(x))dx\right]$(8)

The MADM represents the solution as an infinite series

$y(t) = \sum_{n=0}^\infty y_n(t)$ (9)

The nonlinear operator is decomposed as

$Ny = F(y(x)) = \sum_{n=0}^\infty A_n(y(x))$ (10)

For the nonlinear function $Ny=F(y)$ the first Adomian polynomials are given by

$A_0 = F(y_0)$

$$A_1 = y_1F^{(1)}(y_0),$$
$$A_2 = y_2F^{(1)}(y_0) + \frac{1}{2}{y_1}^2F^{(2)}(y_0),$$
$$A_3 = y_3F^{(1)}(y_0) + y_1y_2F^{(2)}(y_0) + \frac{1}{3}{y_1}^3F^{(3)}(y_0),$$
$$\vdots$$

Substituting Eqns. (9) and (10) into Eqn. (8) we get

$$M\left[\sum_{n=0}^\infty y_n\right] = \sum_{k=0}^{n-1} u^{-k}y^k(0) + \frac{1}{u^\alpha}M[g(t)] + \frac{1}{u^\alpha}M\left[p(t)\sum_{n=0}^\infty y_n\right]$$

$+\frac{\lambda}{u^\alpha}M\left[\int_0^t k(t,x)\sum_{n=0}^\infty A_n(y)\,dx\right]$(11)

Comparing both sides of Eqn. (11) gives the following iterative algorithm:

$M[y_0(t)] = \sum_{k=0}^{n-1} u^{-k}y^k(0) + \frac{1}{u^\alpha}M[g(t)]$,(12)

$M[y_1(t)] = \frac{1}{u^\alpha}M[p(t)y_0] + \frac{\lambda}{u^\alpha}M\left[\int_0^t k(t,x)A_0(y(x))dx\right]$(13)

\

$M[y_2(t)] = \frac{1}{u^\alpha}M[p(t)y_1] + \frac{\lambda}{u^\alpha}M\left[\int_0^t k(t,x)A_1(y(x))dx\right]$(14)

In general, the recursive relation is given by

$$M[y_{n+1}(t)] = \frac{1}{u^\alpha}M[p(t)y_n] + \frac{\lambda}{u^\alpha}M\left[\int_0^t k(t,x)A_n(y(x))dx\right] \quad (15)$$

Applying inverse Mahgoub transform to (12)-(15) we get

$$y_0(t) = M^{-1}\left[\sum_{k=0}^{n-1} u^{-k} y^k(0) + \frac{1}{u^\alpha} M[g(t)]\right]$$

$$y_1(t) = M^{-1}\left[\frac{1}{u^\alpha} M[p(t)y_0]\right]$$

$$+M^{-1}\left[\frac{\lambda}{u^\alpha} M\left[\int_0^t k(t,x)A_0(y(x))dx\right]\right]$$

$$\vdots$$

$$y_{n+1}(t) = M^{-1}\left[\frac{1}{u^\alpha} M[p(t)y_n]\right]$$

$$+M^{-1}\left[\frac{\lambda}{u^\alpha} M\left[\int_0^t k(t,x)A_n(y(x))dx\right]\right]$$

4. NUMERICAL EXAMPLES

In order to demonstrate the efficiency of the MahgoubAdomian decomposition method for solving Nonlinear Volterraintegro-differential equations of fractional order, we present some examples.

4.1. *Example*

Consider the nonlinear Fractional Volterraintegro-differential equation

$$^cD^\alpha y(t) = -1 + \int_0^t [y(x)]^2\, dx, \qquad (16)$$

subject to initial condition $y(0) = 0$ (17)

Exact solution of Eqn. (16) when $\alpha = 1$ is

$$y(t) = -t + \frac{t^4}{12} - \frac{t^7}{252} + \frac{t^{10}}{6048} - \frac{t^{13}}{157248}$$

Applying Mahgoub transform on both sides of Eqn.(16), we have

$$M[\,^cD^\alpha y(t)] = M[-1] + M\left[\int_0^t [y(x)]^2\, dx\right],$$

Using Theorem 2.6 and the initial condition (17),we get

$$M[y(t)] = \frac{-1}{u^\alpha} + \frac{1}{u^\alpha} M\left[\int_0^t [y(x)]^2\, dx\right]$$

Assuming an infinite series solution of the form (9) and (10) in the above equation, we have

$$M\left[\sum_{n=0}^\infty y_n(t)\right] = \frac{-1}{u^\alpha} + \frac{1}{u^\alpha} M\left[\int_0^t \sum_{n=0}^\infty A_n\, dx\right] (18)$$

Comparing both sides of Eqn. (18), we have the following relation

$$M[y_0(t)] = \frac{-1}{u^\alpha}$$

$$M[y_1(t)] = \frac{1}{u^\alpha} M\left[\int_0^t A_0\, dx\right]$$

$$M[y_2(t)] = \frac{1}{u^\alpha} M\left[\int_0^t A_1\, dx\right]$$

$$\vdots$$

$$M[y_{n+1}(t)] = \frac{1}{u^\alpha} M\left[\int_0^t A_n\, dx\right]$$

where the Adomian polynomials for the nonlinearity $F(y) = y^2$ are

$$A_0 = {y_0}^2,$$
$$A_1 = 2y_0y_1,$$
$$A_2 = 2y_0y_2 + {y_1}^2,$$
$$A_3 = 2y_0y_3 + 2y_1y_2,$$
$$A_4 = 2y_0y_4 + 2y_1y_3 + {y_2}^2. \qquad (19)$$
$$\vdots$$

Using the above recursive relation, the first few terms of the MahgoubAdomian decomposition series are derived as follows:

$$y_0(t) = -\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$y_1(t) = \frac{\Gamma(2\alpha+2)}{(\Gamma^2(1+\alpha))(2\alpha+1)\Gamma(3\alpha+2)} t^{3\alpha+1}$$

$$y_2(t) = -\frac{2\Gamma(2\alpha+2)\Gamma(4\alpha+3)}{\Gamma(1+\alpha)^3(1+2\alpha)(2+4\alpha)\Gamma(3\alpha+2)\Gamma(5\alpha+3)} t^{5\alpha+2}$$

$$\vdots$$

The approximate solution is

$$y = y_0 + y_1 + y_2 + \cdots$$

i.e.,$y = -\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{\Gamma(2\alpha+2)}{\left(\Gamma^2(1+\alpha)\right)(2\alpha+1)\Gamma(3\alpha+2)} t^{3\alpha+1}$

$$-\frac{2\Gamma(2\alpha+2)\Gamma(4\alpha+3)}{\Gamma(1+\alpha)^3(1+2\alpha)(2+4\alpha)\Gamma(3\alpha+2)\Gamma(5\alpha+3)} t^{5\alpha+2} + \cdots$$

The approximate solution obtained by our method for different values of αcorresponding to distinct values of t are presented in Table 1(a). The approximate solution for $\alpha = 1$ has been given in Table 1(b) which shows that the numerical solution is very much close to the exact solution. The obtained numerical results for$\alpha = 0.5, 0.75, 0.95$ and $\alpha = 1$ are summarized in Fig. 1.

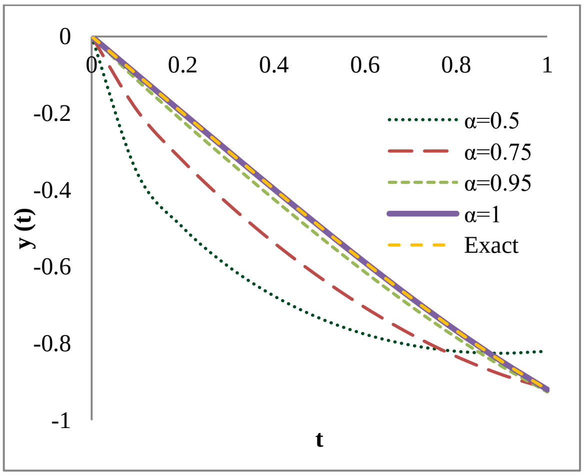
t	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$
0.1	-0.355616	-0.193382	-0.114492
0.2	-0.497843	-0.324393	-0.221008
0.3	-0.599583	-0.437284	-0.324196
0.4	-0.676424	-0.537711	-0.42445
0.5	-0.734276	-0.627400	-0.521437
0.6	-0.776295	-0.706785	-0.614466
0.7	-0.804576	-0.775782	-0.702614
0.8	-0.820682	-0.834122	-0.784816
0.9	-0.825784	-0.881517	-0.859922
1.0	-0.820613	-0.917743	-0.926766

Table 1 (a). Numerical Solution of Eqn. (16) for $\alpha = 0.5, 0.75$ and 0.95

t	$\alpha = 1$		
	Exact	**MADM**	**Error**
0.1	-0.099992	-0.099992	7.580354e-18
0.2	-0.199867	-0.199867	5.218342e-15
0.3	-0.299326	-0.299326	1.013988e-12
0.4	-0.397873	-0.397873	4.267710e-11
0.5	-0.494823	-0.494823	7.762917e-10
0.6	-0.589310	-0.589310	8.305793e-09
0.7	-0.680314	-0.680314	6.161542e-08
0.8	-0.766681	-0.766681	3.496107e-07
0.9	-0.847167	-0.847165	1.616469e-06
1.0	-0.920476	-0.920470	6.359381e-06

Table 1(b). Exact and Approximate solution of Eqn. (16) for $\alpha = 1$

Fig. 1: The approximate and Exact solution of Eqn. (16) for different value of α



4.2. *Example*

Consider the nonlinear Fractional Volterraintegro-differential equation

$$ {}^{c}D^{3/2}y(t) = \frac{8\sqrt{\pi}}{\Gamma^2\left(\frac{1}{2}\right)} t^{3/2} - \frac{15}{56}t^8 + \int_0^t (t+x)[y(x)]^2\,dx, $$

(20)

subject to the initial condition $y(0) = y'(0) = 0$(21)

Exact solution of Eqn. (20) is $y(t) = t^3$

Applying Mahgoub transform on both sides of Eqn. (20), we have

$$ M[\,{}^{c}D^{3/2}y(t)] = M\left(\frac{8\sqrt{\pi}}{\Gamma^2\left(\frac{1}{2}\right)} t^{3/2}\right) - M\left(\frac{15}{56}t^8\right) + M\left(\int_0^t (t+x)[y(x)]^2\,dx\right), $$

Using Theorem 2.6 and the initial condition (21), we get

$$ u^{3/2}M[y(t)] = \frac{8\sqrt{\pi}\,\Gamma\left(\frac{5}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)u^{3/2}} - \frac{15}{56}\frac{\Gamma 9}{u^8} $$
$$ +M\left(\int_0^t (t+x)[y(x)]^2\,dx\right) \qquad (22) $$

Assuming an infinite series solution of the form Eqn. (9) and Eqn. (10) in the above equation, we have

$$M\left[\sum_{n=0}^{\infty} y_n(t)\right] = \frac{8\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)u^3} - \frac{15}{56}\frac{\Gamma 9}{u^{\frac{19}{2}}}$$
$$+\frac{1}{u^{3/2}} M\left(\int_0^t (t+x)\left[\sum_{n=0}^{\infty} A_n\right] dx\right) \qquad (23)$$

Comparing both sides of (23), we have the following relation

$$M[y_0(t)] = \frac{8\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)u^3} - \frac{15}{56}\frac{\Gamma 9}{u^{19/2}}$$
$$M[y_1(t)] = \frac{1}{u^{3/2}} M\left(\int_0^t (t+x)A_0\, dx\right)$$
$$\vdots$$
$$M[y_{n+1}(t)] = \frac{1}{u^{3/2}} M\left(\int_0^t (t+x)A_n\, dx\right)$$

where the Adomian polynomials for the nonlinearity $F(y) = y^2$ are given in Eqn. (19)

Using the above recursive relation, the first few terms of the MahgoubAdomian decomposition series are derived as follows:

$$y_0(t) = \frac{8\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)\Gamma(4)} t^3 - \frac{15}{56}\frac{\Gamma 9}{\Gamma\left(\frac{21}{2}\right)} t^{19/2}$$
$$y_1(t) = \frac{64\pi\Gamma^2\left(\frac{5}{2}\right)\Gamma(9)}{7\Gamma^4\left(\frac{1}{2}\right)\Gamma^2(4)\Gamma\left(\frac{21}{2}\right)} t^{\frac{19}{2}}$$
$$+\left(\frac{15}{56}\right)^2 \frac{\Gamma^2 9}{\Gamma^2\left(\frac{21}{2}\right)}\frac{\Gamma 22}{20\Gamma\left(\frac{45}{2}\right)} t^{\frac{43}{2}}$$
$$-\frac{20}{63}\frac{\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)\Gamma(9)\Gamma\left(\frac{31}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)\Gamma(4)\Gamma\left(\frac{21}{2}\right)\Gamma(17)} t^{16}$$
$$+\frac{64\pi\Gamma^2\left(\frac{5}{2}\right)\Gamma(9)}{8\Gamma^4\left(\frac{1}{2}\right)\Gamma^2(4)\Gamma\left(\frac{21}{2}\right)} t^{\frac{19}{2}}$$
$$+\left(\frac{15}{56}\right)^2 \frac{\Gamma^2 9}{\Gamma^2\left(\frac{21}{2}\right)}\frac{\Gamma 22}{21\Gamma\left(\frac{47}{2}\right)} t^{\frac{45}{2}}$$
$$-\frac{20}{63}\frac{\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)\Gamma(9)\Gamma\left(\frac{29}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)\Gamma(4)\Gamma\left(\frac{21}{2}\right)\Gamma(16)} t^{15}$$
$$\vdots$$

The approximate solution is

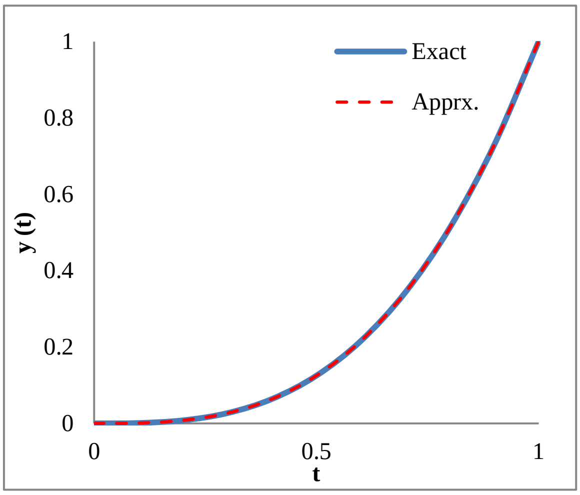
$$y(t) = \frac{8\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)\Gamma(4)} t^3 - \frac{15}{56}\frac{\Gamma 9}{\Gamma\left(\frac{21}{2}\right)} t^{\frac{19}{2}}$$
$$+\frac{64\pi\Gamma^2\left(\frac{5}{2}\right)\Gamma(9)}{7\Gamma^4\left(\frac{1}{2}\right)\Gamma^2(4)\Gamma\left(\frac{21}{2}\right)} t^{\frac{19}{2}} + \left(\frac{15}{56}\right)^2 \frac{\Gamma^2 9}{\Gamma^2\left(\frac{21}{2}\right)}\frac{\Gamma 22}{20\Gamma\left(\frac{45}{2}\right)} t^{\frac{43}{2}}$$
$$-\frac{20}{63}\frac{\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)\Gamma(9)\Gamma\left(\frac{31}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)\Gamma(4)\Gamma\left(\frac{21}{2}\right)\Gamma(17)} t^{16} + \cdots$$

The exact solution and our approximate solution obtained by our method corresponding to distinct values of t are presented in Table 2. Fig. 2 shows the approximate and exact solution of Eqn. (20)

Table 2: Numerical Solution of Eqn. (20)

t	Exact	Approx	Error
0.1	0.001000	0.001000	2.719110e-20
0.2	0.008000	0.008000	9.650350e-16
0.3	0.027000	0.027000	4.550033e-13
0.4	0.064000	0.064000	3.647436e-11
0.5	0.125000	0.125000	1.105616e-09
0.6	0.216000	0.216000	1.809660e-08
0.7	0.343000	0.343000	1.934521e-07
0.8	0.512000	0.511998	1.513145e-06
0.9	0.729000	0.728991	9.314557e-06
1.0	1.000000	0.999953	4.701856e-05

Fig. 2: The approximate and exact solution of Eqn. (20)



5. CONCLUSION

In this paper, MahgoubAdomian decomposition method has been successfully applied to find the approximate solution of nonlinear Volterraintegro-differential equation of fractional order. This method gives good and accurate solution when compared with exact solution. Therefore, our proposed method is very effective and powerful.

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