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A Mathematical Model for Harmonic Analysis of an Inclined 3-DOF Cantilever Beam

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ABSTRACT

Frequency response analysis is commonly used in the engineering industry to determine the structural integrity against harmonically varying loads. For simple structures, hand calculations can be used but with an increase in complexity, the hand calculations become infeasible and recourse is made to solution using numerical methods typically with the help of commercially available software. But even the most famous of the softwares is a black box whose internal working algorithms cannot be verified by the user for complicated problems. Hence it is necessary to verify the result, if possible, by some approximate hand calculations that provide a sanity check. In this paper, a mathematical model is derived to validate the results of the frequency response analysis of the actuator. The analytical model developed, apart from validating the frequency response analysis also provides a means for developing the design of the actuator. Further, the mathematical case study provided gives some interesting insights into the vibratory behavior of the structure.

Keywords—component; formatting; style; styling; insert (key words)

I. INTRODUCTION

Many structures like aerospace actuators, aircrafts, rotating machinery, etc are required to be resistant to harmonic vibratory excitation. Resonance of the structure is usually not accepted, and the structures enduring resonance are usually condemned to be redesigned. The resonant frequencies are found by modal analysis of the structure and these frequencies are generally required to be avoided in the harmonic loading. It must be noted that not all modes are considered but all modes within a band of frequencies. The usual approach is to avoid the natural frequencies of all the modes with in this range of frequencies. But it must be noted that the bodes plot does not show peaks at all resonant frequencies leading to questioning of the correctness of the results obtained.

The equation of motion for 1-DOF vibration problems is [1]

$m\ddot{x}+C\dot{x}+Kx=f(t)$ (1)

Real life structures are continuous systems, but most structures are approximated as multi-DOF semi discrete systems which can be evaluated by finite element analysis using computers. For multi-degree of freedom systems, the governing equation of motion becomes

$[m]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f(t)\}$ (2)

The solution for equation (2) is [2] x = CF(t)+PI(t)

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the complimentary function (CF) represents the transient response and the particular integral (PI) represents the steady state solution. In engineering problems often, only the steady state vibration is considered for evaluation of the structural integrity. For a system under harmonic loading the (2) becomes:

$[m]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f\}sin\omega t$ (3)

For these type of problems, which are often encountered in the industry, it is useful to approximate the problem by dividing it into modal responses. Modes are those values of {x} which satisfy the homogeneous part of the equation (3):

$[m]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=0$ (4)

It is found that for the range of values of [c] found in most industrial problems, the solution to (4) is almost independent of the [C]. Thus, a solution of acceptable accuracy can be obtained by simplifying the equation to:

$[m]\{x\}+[K]\{x\}=0$ (5)

Equation (5) can be solved as an Eigen value problem [2]. The Eigen values are called the natural frequencies and the eigen vectors obtained are called the mode-shapes [3]. Each of the modes behaves as an independent oscillator and the response of the system is the superposition of responses of all the modes. As many modes as the number of independent DOF's can be found but only a few modes are usually of interest in the vibration problems.

The particular integral for equation (2) can be expressed as [4]:

$\{x\} = \{X_0 \sin(\omega t - p)\}$ (6)

Where p is the phase difference between the applied force and the resultant displacement. Expressing equation (2) in modal coordinates we get, [5]

$$M_{0i}\,\ddot{\varepsilon} + C_{0i}\,\dot{\varepsilon} + K_{0i}\,\varepsilon = \{\varphi_{iT}\}\{f\} sin\omega t$$

Where is the ith mode shape vector and the response of the ith mode is

$\{xi\} = \{\varphi i\}\varepsilon i$

The response of the structure can be split into the individual modal responses. The structural responses can be expressed as [5]:

$$x = \sum_{i=1}^{N} \varphi_i.\, \varepsilon_i.\, \sin(\omega t - P_i)$$

Where,

$\varphi 1$ is the ith mode shape vector and is the ith modal response magnitude. N is the number of significant modes. The value of is obtained from the relation:

$$\varepsilon_i = \frac{\varphi_i^T f}{\sqrt{(K_i - m_i\omega^2)^2 + (c_i\omega)^2}} \qquad (1)$$

Which can be expressed as:

$$\varepsilon_i = \frac{\frac{\varphi_i^T f}{k_i}}{\sqrt{\left(1-r_i^2\right)^2 + (2\tau_i r_i)^2}} \qquad (2)$$

Where,

$r_i = \omega/\omega_i$ is the frequency ratio of the ith mode

$\tau_i = c_i/C$ 晦$_i$ is the frequency ratio of the ith mode

ci is the normalized damping constant for the ith mode

cc_i is the critical damping coefficient for the ith mode = $2m_i\omega_i$

ωi is the natural frequency of the ith mode

ω is the frequency of excitation

mi is the normalized modal mass of the ith mode

Pi is the phase angle lag of the ith mode

f is the load vector

ki is the normalized modal stiffness of the ith mode

In case constant damping ratio τ is assumed, $\tau i = \tau$

It must be noted that any load of any orientation at any location and at any frequency excites all the modes but different loads excite different modes to varying extents leading to different deformed shapes.

II. INCLINED CANTILEVER WITH 3 DEGREES OF FREEDOM

The cantilever considered for calculation is inclined at an angle to the direction of the load as shown in figure2. The load Fx is along the x-axis with zero y and z components but since the beam is inclined to the coordinate system, the loads are transformed to a new coordinate system with x-axis along the axis of the beam.

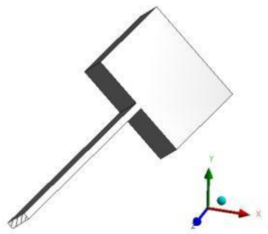

Figure 1 Beam with end load

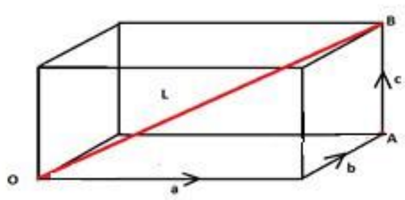

Figure 2 beam with a coordinate system

Loads in the transformed co ordinates

Figure 3 forces on the beam

The loads in the new coordinate system are found by transformation and are as follows: Taking top view,

$$Fx' = \frac{Fx.a}{\sqrt{a^2 + b^2 + c^2}}$$

$$Fy' = \frac{Fx.b}{\sqrt{a^2 + b^2 + c^2}}$$

$$Fx' = \frac{Fx.c}{\sqrt{a^2 + b^2 + c^2}}$$

Modal Analysis:

The stiffness of the system in the three directions of the transformed coordinate system are found from beam theory [6]:

$$K_x = \frac{t.w.E}{L}$$

$$K_y = \frac{E.t.w^3}{4L^3}$$

$$K_z = \frac{E.w.t^3}{4L^3}$$

The mass at the end of the beam is m. Hence the natural frequencies of the system are:

$\omega_1 = \sqrt{\frac{K_z}{m}}$, $\omega_2 = \sqrt{\frac{K_y}{m}}$, and $\omega_3 = \sqrt{\frac{K_x}{m}}$, assuming Kx>Ky>Kz

The mode shapes are $\varphi_1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$, $\varphi_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$, $\varphi_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

The force vector is f= $\begin{Bmatrix} Fx' \\ Fy' \\ Fz' \end{Bmatrix}$

Harmonic Analysis:
Eqn(2) can be expanded and written as:

$$\varphi_1 \varepsilon_1 = \frac{\varphi_1^T f / k_1}{\sqrt{(1 - r_1^2)^2 + (2\tau_1 r_1)^2}}$$

$$u_z = \varphi_1 \varepsilon_1 = \frac{Fz'/K_z}{\sqrt{(1-r_1^2)^2+(2\tau_1 r_1)^2}} = \frac{\delta_{st1}}{\sqrt{(1-r_1^2)^2+(2\tau_1 r_1)^2}}$$

Since $K_1 = K_z$

$$u_y = \varphi_2 \varepsilon_2 = \frac{Fy'/K_y}{\sqrt{(1-r_2^2)^2+(2\tau_2 r_2)^2}} = \frac{\delta_{st2}}{\sqrt{(1-r_2^2)^2+(2\tau_2 r_2)^2}}$$

$$u_x = \varphi_3 \varepsilon_3 = \frac{Fx'/K_x}{\sqrt{(1-r_3^2)^2+(2\tau_3 r_3)^2}} = \frac{\delta_{st3}}{\sqrt{(1-r_3^2)^2+(2\tau_3 r_3)^2}}$$

The response of an individual mode can be represented as $\varphi 1. \varepsilon i$. If the mode shape vector is scaled by a factor n, then the new mode-shape vector is nφ1. The new modal response can be written as

$$n\varphi_i . \varepsilon_i(n\varphi_i) = n\varphi_i \frac{n\varphi_i^T f / k_i}{\sqrt{(1-r_i^2)^2+(2\tau_i r_i)^2}}$$

$$= n\varphi_i \frac{n\varphi_i^T f / n\varphi_i^T K . n\varphi_i}{\sqrt{(1-r_i^2)^2+(2\tau_i r_i)^2}}$$ where, K is the stiffness matrix

$$= n\varphi_i \frac{n\varphi_i^T f / n\varphi_i^T K . n\varphi_i}{\sqrt{(1-r_i^2)^2+(2\tau_i r_i)^2}}$$

$$= n\varphi_i \frac{1}{n} \frac{\varphi_i^T f / \varphi_i^T K . \varphi_i}{\sqrt{(1-r_i^2)^2+(2\tau_i r_i)^2}}$$

$$= n\varphi_i \frac{1}{n} \frac{\varphi_i^T f / k_i}{\sqrt{(1-r_i^2)^2+(2\tau_i r_i)^2}}$$

$$= \varphi_i \frac{\varphi_i^T f / k_i}{\sqrt{(1-r_i^2)^2+(2\tau_i r_i)^2}}$$

$$= \varphi_i . \varepsilon_i$$

Thus, the modal responses are not affected by the scaling of the mode shape vector. Hence although the modal vector gets scaled from one software to another, the dynamic response of the modes and that of the structure itself remains unaffected by the scaling.

The cantilever beam considered is made of steel and has a length of 40 mm, depth 20 mm, width 10 mm having a mass of 40 Kg at its free end. An acceleration load of 0.5 G is applied to the structure in the x-direction.

III. RESULTS

The displacement and the maximum principal stress were tracked against the exciting frequency and the following bode plots were obtained:

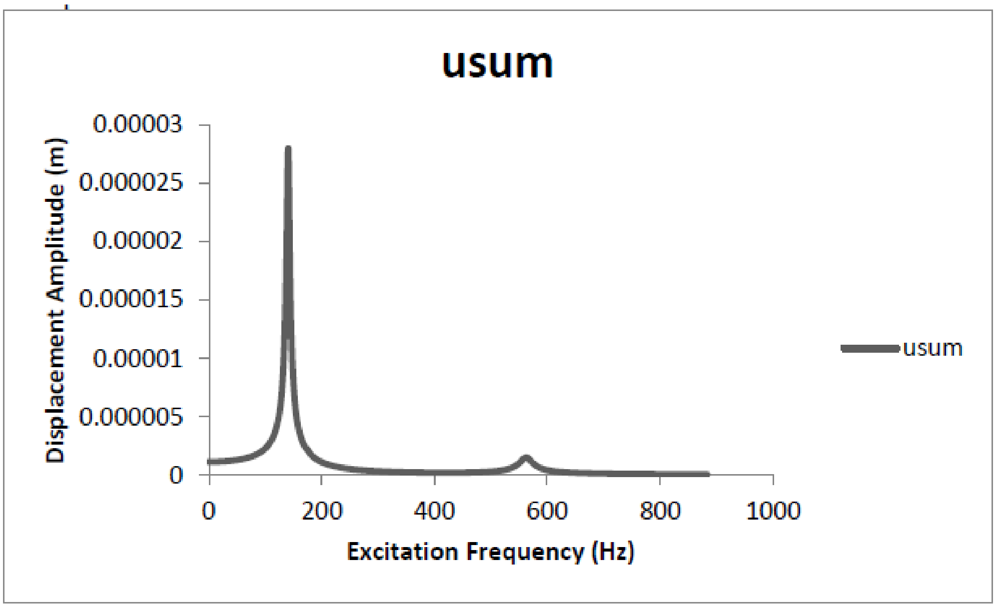

Figure 4 Displacement vector sum Vs frequency

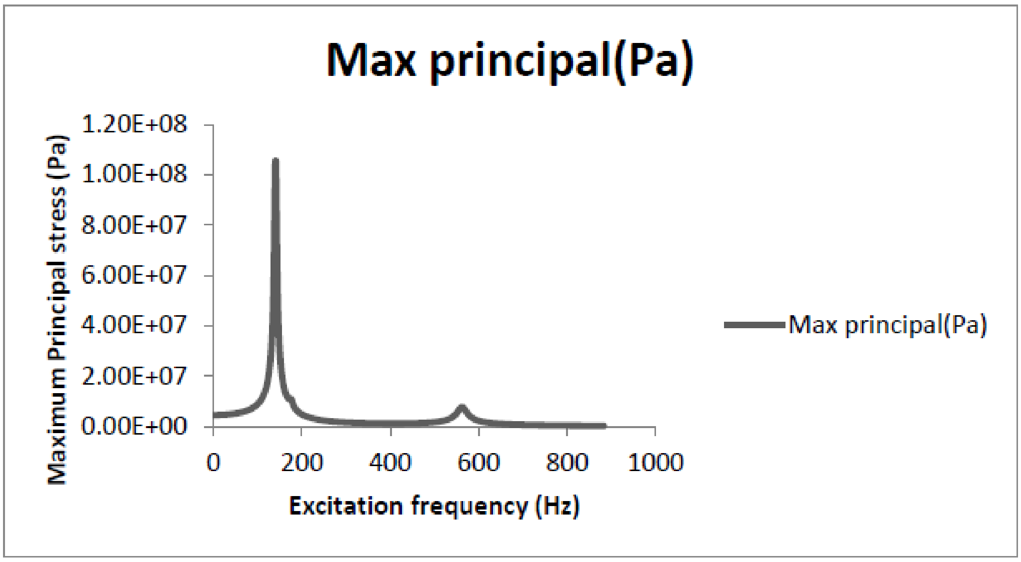

Figure 5 Maximum Principal Stress Vs frequency

The natural frequencies obtained were 140.67 Hz, 175.84 Hz and 562.7 Hz.

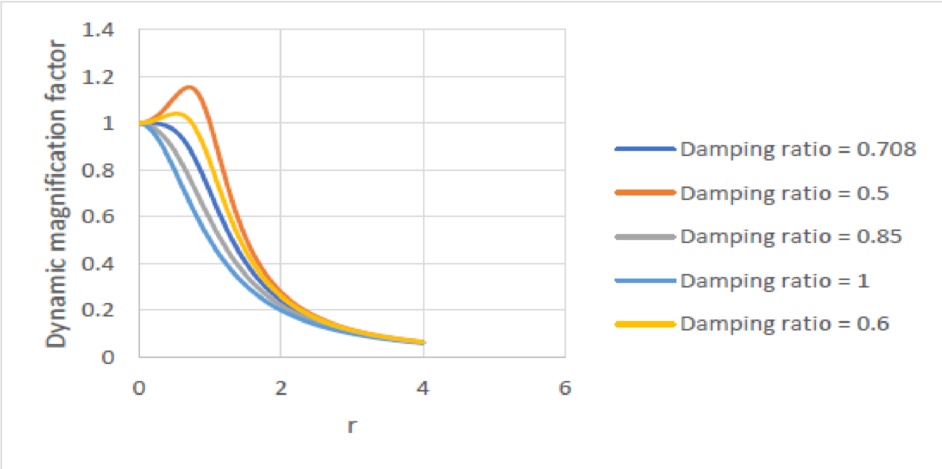

Figure 6 Dynamic magnification factor vs damping ratio

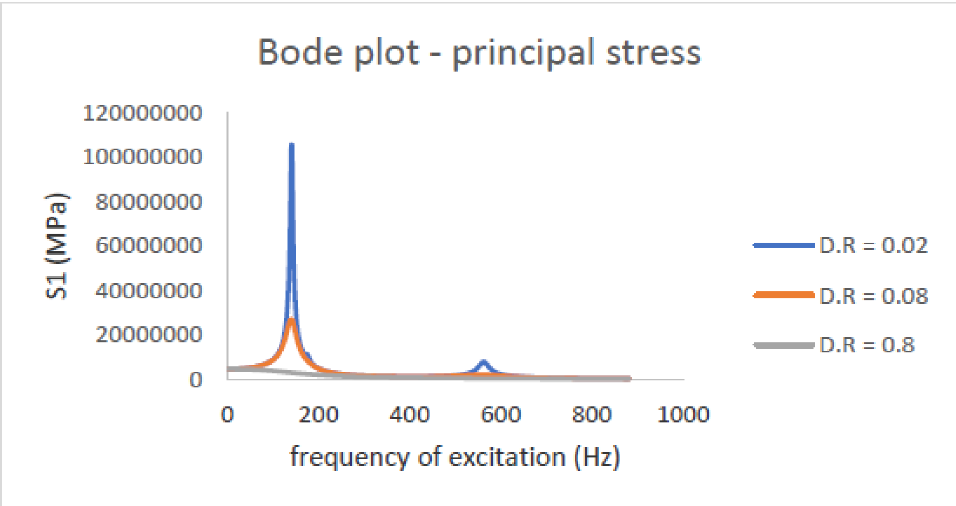

Figure 7 Maximum Principal Stress Vs frequency for different damping ratios

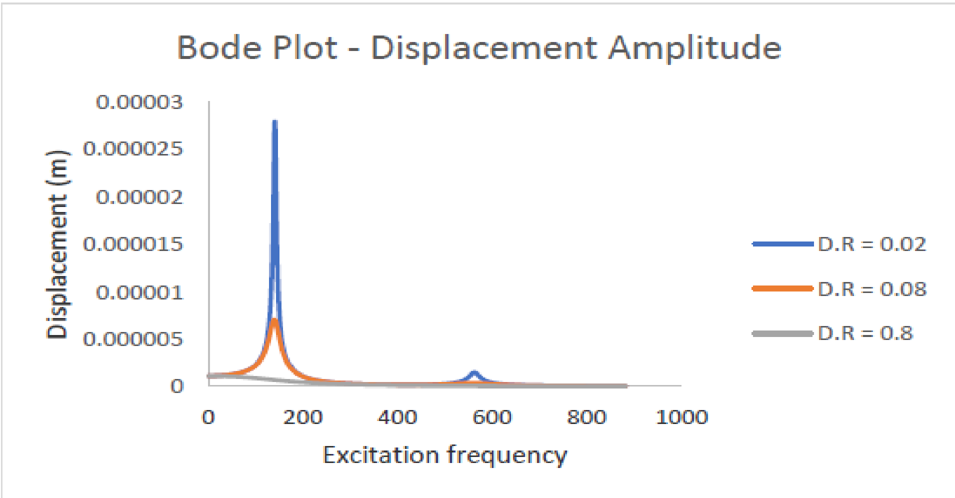

Figure 8 Maximum displacement Vs frequency for different damping ratios

It can also be noted that that with an increase in the damping ratio, the peak displacement shifts further and further away from the resonant frequency.

IV. DISCUSSION

It must be noted that the bode plot does not exhibit peaks at all resonant frequencies as shown in fig.4. The response of some of the modes is weak leading to no peaks or weak peaks at the resonant frequency of the mode. This behaviour becomes more prominent at higher levels of damping as shown in fig8.

For 1-dof system, from the normalized bode plot is shown in fig.6, it can be seen that for damping ratio more than 0.5, the response at the resonant frequency is less than the static displacement and for damping ratio more than √2, there are no peaks in the bode plot and the dynamic displacement is always lesser than static displacement. The modes also follow the same behaviour.The induced stress is less than the yield stress.With an increasing damping ratio the smaller peaks in the bode plot begin to disappear and in fact, for damping ratio more than 0.707, there are no peaks in the bode plot and the maximum displacement occurs for the zero frequency and is equal to the static displacement. The peak for the second resonant frequency is present in the maximum principal stress vs frequency plot but not in the displacement vs frequency plot.

V. DISCUSSION

The structural performance of a 3-DOF beam with mass was modelled analytically and the structural performance was studied parametrically. It was proven that the scaling of the mode shape vector at the time of the mode extraction does not affect the evaluation of the structural performance of the system. It was also found that the bode plot for stress can have more peaks than the bode plot for displacement. Also, the bode plot does not exhibit peaks at all the resonant frequencies. The matching of the natural frequency with the exciting frequency does not cause failure of the system and it is also found that the damping causes some of the peaks in the bode plot to disappear and beyond a damping ratio of 0.707, no peaks are seen in the bode plot.

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