Triangular Graphs, Tridi Graphs and Related Aspects

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Abstract: In this paper, we define different types of triangular graphs and tridi graphs. We prove the relation between the number of vertices and the number of rectangular regions in the geometric duals of greatest triangular graphs and some related results. We give a direct formula for the number of triangular and rectangular regions in the geometric dual of Triangular Graph. This paper defines the * outerplanar graph and prove the necessary and sufficient condition for * outerplanarity of graphs"

Index Terms: Triangular Graph, *outerplanar graph, tridi graph.

1. INTRODUCTION

It is no coincidence that different mathematicians have been discovered graph theory many times independently. It may quite properly be regarded as an important area of applied mathematics. Euler became a father of graph theory, when in 1936 he solved a famous unsolved problem of his day, called the Konigsberg Bridge Problem. In 1847, Kirchhoff developed the theory of trees which helps to solve problems of an electrical network. In 1936 the psychologist Lewin proposed that the life space of an individual represented by a planar graph. There are several other criteria for identifying planarity that have been discovered in the original work of Kuratowski. Tutte developed an important algorithm for drawing a graph in a plane. Whitney expressed planarity in terms of the existence of dual graphs. We find some interesting properties of Triangular graphs which will play very significant role in the further development of applications of graph theory. Covering every aspect of graphs would take hundreds of pages. This article provides a quick overview of graphs, Triangular graphs and outerplanar graphs. It introduces a few techniques for dealing with graphs, and explores some interesting problems. This article will hopefully shed some light on the beauty of Triangular graphs and related aspects.

In this section, we present a brief survey of those results of graph theory, which we shall need later. The reader is referred to [4, 5, 6, 7] for a fuller treatment of the subject.

1.1. Graphs

A graph G is an ordered pair (V (G), E (G)) where i) V(G) is a non empty finite set of elements , known as vertices. V (G) is known as vertex set. ii) E(G) is a family of unordered pairs (not necessarily distinct) of elements of V, known as edges of G.

A walk of a graph is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. Vertices of graph with which a walk begins and ends, are called its terminal vertices. A walk, in which terminal vertices are same, is called as the closed walk .Otherwise open walk. A graph G is said to be the connected graph if there exists at least one walk between every pair of vertices in G .Otherwise graph G is disconnected. The vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected. A connected graph is said to be 2- connected if its vertex connectivity is two. [5]

1.2. Planar graph

A graph G is a planar graph if it is possible to represent it in the plane such that no two edges of the graph intersect except possibly at a vertex to which they are both incident. Any such drawing of planar graph G in a plane is a planar embedding of G. The degree of the region is the number of edges in a closed walk that encloses it. The region formed by three edges is known as triangular region. The region formed by four edges is known as rectangular region. [5, 6]

Theorem 1.1: If a connected planar graph G with n vertices, m edges has f regions or faces, then $n - m + f = 2$.

1.3. Triangular graph:

A graph is said to be a Maximal Planar Graph if the graph becomes non-planar when any two nonadjacent vertices in it are joined by an edge. A maximal planar graph is necessarily a connected graph. Every graph is a spanning subgraph of a maximal planar graph. A planar graph is a maximal planar graph if and only if the rank of every region of that graph is three. Therefore every maximal planar graph has a straight line representation. The maximal planar graph is also known as triangular graph. [6]

1.4. Outerplanar graph

A planar graph is said to be an Outerplanar graph if it can be embedded in the plane so that all its vertices lie on the same region or face, that region or face may be exterior or interior region. It is first named and studied by Chartrand and Harary in 1967. [1]

1.5. Maximal outerplanar graph

An Outerplanar graph is said to be Maximal Outerplanar graph if it loses its outerplanarity when any two non adjacent vertices are joined by an edge. [2]

Theorem 1.2: Every maximal Outerplanar graph G with n vertices has (i) 2n-3 edges, (ii) At least two vertices of degree 2. [5]

1.6. Geometric dual

Let G be a plane graph with n Regions or faces say $R_1, R_2, R_3, \ldots R_n$. Let us place points (say vertices) $V_1, V_2, V_3, \ldots V_n$, one in each of the regions. Let us join these vertices V_i according to the following procedure.

i) If two regions R_i and R_j are adjacent then draw a line joining vertices V_i and V_j that intersect the common edge between R_i and R_j exactly once.

ii) If there are two or more edges common between R_i and R_j , then draw one line between vertices V_i and V_j for each of the common edges.

iii) For an edge 'e' lying entirely in one region say R_i, draw a self loop at pendant vertex V_i intersecting e exactly once.

By this procedure, we obtain a new graph G* consisting of $V_1, V_2, V_3, \ldots V_n$ vertices and edges joining these vertices. Such a graph G* is called a geometric dual of G (a dual of G). [5]

1.7. * isomorphic graphs

Two graphs G_1 and G_2, said to be * isomorphic graphs if their duals are isomorphic. Every graph is * isomorphic to itself. [3]

1.8. HB graph

A region or face R of a planar graph is said to be a pivot region of graph if all other regions of graph are adjacent to R. A planar graph is said to be HB graph if it has a pivot region. [4]

2. TYPES OF TRIANGULAR GRAPHS

Depending upon the nature of degree sequence of triangular, there are three different types of triangular graphs as given below.

2.1. Greatest triangular graph

A triangular graph G on $n \geq 4$ vertices is said to be the greatest triangular graph if the degree sequence of graph G is n – 1, n – 1, 4, 4 . . . 4, 4, 3, 3. The greatest triangular graph on n vertices is denoted by GT_n. In the greatest triangular graph, there must be at least two vertices of degree 4 and exactly two vertices of degree three. The degree sequence of the greatest triangular graph on six vertices (GT_6) is 5, 5, 4, 4, 3, 3.

2.2. Top triangular graph

A triangular graph G on $n \geq 6$ vertices is said to be the top triangular graph if the degree sequence of graph G is n – 2, n – 2, 4, 4 . . . 4, 4. The top triangular graph on n vertices is denoted by TT_n. Every TT_n has at least five vertices of degree 4. The degree sequence of the top triangular graph on 8 vertices (TT_8) is 6, 6, 4, 4, 4, 4, 4, 4.

2.3. Regular triangular graph

A triangular graph, in which degree of each vertex is same, is called regular triangular graph. Alternately, a graph which is regular as well as triangular is called regular triangular graph. The regular triangular graph on n vertices is denoted by RT_n. The degree sequence of a regular triangular graph on 6 vertices (RT_6) is 4, 4, 4, 4, 4, 4.

Theorem 2.1: The geometric dual of a Triangular graph on $n \geq 4$ vertices is a simple graph.

Proof: Let G be a Triangular graph on $n \geq 4$ vertices. The degree of every region interior as well as exterior of G is 3. Every triangular region of graph G is adjacent to three different triangular regions and there is no any pendent vertex. So there are no any parallel edges or loops in the geometric dual of G. Hence the geometric dual of G is a simple graph.

Theorem 2.2: The geometric dual of the greatest Triangular graph on n vertices, where $n \geq 6$, has n-4 rectangular regions.

Proof: Here we use the principal of mathematical induction to prove this result. We assume that the result is true for n=k vertices. That is, the geometric dual of the greatest triangular graph G on k vertices, where $k \geq 6$, has k-4 rectangular regions. Let G_1 be the triangular graph with k+1 vertices. If we add one vertex to graph G, then we get two more triangular regions in G. Therefore G_1^* has one more rectangular region than G^*. So G_1^* has (k-4) +1 =(k+1) – 4 rectangular regions. Thus by the principal of mathematical induction, we get the required result.

Theorem 2.3: If G is a Triangular graph on n vertices, where $n \geq 5$, then G has 2(n-2) regions.

Proof: By Euler's Formula for Plane Graphs, we get n - m + f = 2 . We know that a triangular graph on n vertices has 3n – 6 numbers of edges. So m=3n-6. Therefore we have $n - (3n-6) + f = 2 \Rightarrow f = 2(n-2)$. Thus G has 2(n-2) regions.

Theorem 2.4: If G is the greatest triangular graph on n vertices, where $n \geq 5$, then f = n + s, where f = Number of regions or faces in G, s = Number of rectangular regions in G^*.

Proof: Let G be a triangular graph on $n \geq 5$ vertices with m edges and f faces or regions. By theorem f = 2n-4 , m=3n-6 and s = n-4. By Euler's theorem, we have

$$n - m + f = 2$$

$$\Rightarrow f = 2 - n + m$$

$\Rightarrow$ f =2-n + 3n-6

$\Rightarrow$ f = 2 n - 4 = n + (n - 4) = n + s

$\Rightarrow$ f = n + s.

Theorem 2.5: If T is the top triangular graph on n vertices, e edges, m regions, d total degree and graph T* has s number of rectangular regions, then e = 3s, m = 2s and d = 6s.

Proof: Let T be the triangular graph on n vertices, e edges and m regions. The total degree of graph T is d. The graph T* has s rectangular regions. T* has n – 2 rectangular regions. Therefore s = n – 2. We know that the triangular graph on n vertices has 3n – 6 edges. So e = 3n - 6 = 3 (n - 2) = 3s.

As triangular graph has 2n – 4 rectangular regions. Hence m = 2n – 4 = 2 (n - 2) = 2s.

As every region of T is triangular, the rank of each region of T is 3. Therefore the total degree of T is d = 3 (2n - 4) = 6 (n - 2) = 6s. Hence the proof.

Theorem 2.6: Every triangular graph on n ≥ 5 vertices is not HB graph.

Proof: Let G be a triangular graph on n ≥ 5 vertices. By theorem 2.3, G has 2 (n-2) triangular regions. Graph G has at least 6 Triangular regions. So there exists one triangular region, say T_1, which is adjacent to three triangular regions T_2, T_3, T_4.Therefore remaining triangular regions are not adjacent to T_1. So the triangular graph G has no pivot region. Hence G is not HB graph.

Theorem 2.7: There are exactly four regular triangular graphs.

Proof: Let G be a regular triangular graph on n vertices. Graph G has 3n – 6 edges and total degree of G is 2 (3n - 6). As graph G is regular, so n divides 6n - 12. This implies 6n – 12 = n k, where k is any non negative integer.

$\Rightarrow$ 6n – nk =12 $\Rightarrow$ n (6 - k) = 12

$\Rightarrow$ n = 12/ (6-k) $\Rightarrow$ k = 0, 2, 3, 4 and 5.

For k = 0, 2, 3, 4 and 5 the values of n are 2, 3, 4, 6 and 12 respectively. The triangular graph on two vertices does not exist. Thus there are exactly four regular triangular graphs. The regular triangular graph on three vertices is called cycle graph, four vertices is tetrahedron (The complete graph on four vertices), six vertices is octahedron and twelve vertices is called icosahedrons.

Theorem 2.8: Every regular triangular graph is a regular rank graph. Converse is not true.

Proof: Let G be a regular triangular graph. Every region of G is bounded by three edges, so rank of each region of G is same. All vertices of G are of the same degree. Thus graph G is regular rank graph.

Let C_n be a cycle graph on four or more vertices. Graph C_n is regular rank graph. Graph C_n has only two regions and rank of each region is n, which is different from three. Therefore C_n is not triangular graph. Hence a regular rank graph may or may not be regular triangular graph.

Cube graph and Dodecahedron graph are regular rank graphs but not regular triangular graphs.

3. *TRIANGULAR GRAPH

A planar graph G is said to be *triangular graph if it's geometric dual (G*) is a triangular graph. The complete graph on four vertices is a triangular graph and it's geometric dual is also triangular graph So K_4 is *triangular graph. Moreover the complete graph on three vertices is triangular graph but not *triangular graph.

Theorem 3.1: In the geometric dual of maximal Outerplanar graph on n ≥ 4 vertices, there are at least two pairs of multiple edges.

Proof: Every maximal Outerplanar graph has at least two vertices of degree 2. So there are at least two triangular regions which have two common edges with an exterior region. If two regions have two boundary edges common, then these edges form parallel edges in its geometric dual. Hence it is proved .

Corollary 3.1.1: The geometric dual of maximal Outerplanar graph is not simple graph.

Proof: By theorem 3.1, there are at least two pairs of multiple edges in the geometric dual of maximal Outerplanar graph. Hence geometric dual is not simple.

4. * OUTERPLANAR GRAPH

A planar graph G is said to be * Outerplanar graph if its geometric dual is an Outerplanar graph.A planar graph G is called Absolute * Outerplanar graph if graphs G and G^* are an Outerplanar graphs. A planar graph G is called conditional * Outerplanar graph if G^* is an Outerplanar graph but G is not an Outerplanar graph.

Theorem 4.1: A graph is * Outerplanar graph if and only if it has no subgraph homeomorphic to K_4 or H, where H is the regular connected multigraph on 3 vertices and 6 edges.

Proof: Let G be * outerplanar graph. Therefore G^* is outerplanar graph. Graph G^* has no subgraph that is homeomorphic to K_4 and $K_{2,3}$. The geometric duals of K_4 and $K_{2,3}$ are K_4 and graph H respectively, where H is the regular connected multigraph on 3 vertices with 6 edges. So graph G has no subgraph homeomorphic to K_4 and H. Conversely, if graph G has no subgraph homeomorphic to K_4 and H, then G^* has no subgraph homeomorphic to K_4 and $K_{2,3}$. So G^* is outerplanar graph. Hence G is * outerplanar graph.

5. TRIDI GRAPH

A cycle of length one is called improper cycle. All loops are improper cycles. A connected graph G is said to be Tridi graph if it has improper cycles only and each vertex is adjacent to at most two vertices.

The adjacency matrix of this graph is tri-diagonal matrix. So the graph is known as tri-diagonal graph and the name Tri-di graph comes from tri-diagonal graph. For convenience we use Tridi graph. Tridi graph having n number of vertices and m loops is denoted by $T_{n,m}$ or T (n, m). Therefore, $T_{n,0}$ represents a chain graph on n vertices. $T_{1,m}$ represents a Rose graph on m loops. $T_{2,1}$ represents a tridi graph with two vertices and one loop.

Theorem 5.1 : The geometric dual of Tridi graph $T_{n,0}$ is a $T_{1,n-1}$.

Proof: Let G be a Tridi graph $T_{n,0}$. By definition of Tridi graph, G is a connected graph with n vertices and n-1 edges. So G is a chain graph with n vertices. G has only one region with rank n-1. Therefore the geometric dual of G has one vertex and n-1 loops. Such type of graph is denoted by $T_{1,n-1}$. Thus the geometric dual of Tridi graph $T_{n,0}$ is a $T_{1,n-1}$.

5.1. Complete tridi graph

A Tridi graph having equal number of vertices and loops is called Complete Tridi graph. A Tridi graph may be regular graph.

5.2. Regular tridi graph

A Tridi graph, in which degree of each vertex is same, is called regular tridi graph. It is denoted by $T_{n,n}$. $T_{3,3}$ represents a regular tridi graph with 3 vertices and 3 loops.

5.3. L^* graph

A connected graph G is said to be L^* graph if there exists a vertex v such that d (v) ≥ 3 with at least one loop is incident at v and remaining all vertices are pendent vertices. A L*graph with n number of vertices and m loops is denoted by $L^*_{n,m}$ or L* (n, m). A L* graph $L^*_{n,0}$ is a star graph. L* (3, 2) represents a L* graph with 3 vertices and 2 loops.

5.4. Complete L^* graph

A L^* graph having equal number of vertices and loops, is called Complete L^* graph. It is denoted by L* (n, n).

Theorem 5.2 : The geometric dual of L* (n, m) graph if no loop present inside of other loops, is a L* (m+1, n-1) graph.

Proof: Let G be a L* graph with n vertices and m loops such that no loop present inside of other loops. G has m + 1 regions and m+ n–1 edges. Out of total number of edges, G has n–1 bridges only. Therefore the geometric dual of G say G*, has m+1 vertices and n-1 loops. G* has m pendent vertices and one vertex is of degree (2n + m – 2) ≥ 3. Thus graph G* is L* (m+1, n-1) graph.

Theorem 5.3 : The geometric dual of T (n, m) graph if no loop present inside of other loops, is a L* (m+1, n-1) graph.

Proof: Let G be a Tridi graph with n vertices and m loops such that no loop present inside of other loops. The geometric dual of G say G* has m+1 vertices and m+n-1 edges. So G* has n-1 loops. G* has m pendent vertices and one vertex is of degree (2n + m – 2) ≥ 3. Thus graph G* is L* (m+1, n-1) graph.

6. LB GRAPH

A connected graph having only improper cycles is called LB graph. Thus LB graph has only loops and/ or bridges. LB graph with n vertices and *l* loops is denoted by $L_{n,l}$. Every $L_{n,l}$ graph has n-1+*l* edges.

6.1. Complete LB graph

A LB graph having equal number of vertices and loops is called Complete LB graph.

6.2. Purely LB graph

A LB graph which is neither Tridi graph nor L* graph is called purely LB graph.

Every Tridi graph is a LB graph but converse is not true. Every L* graph is a LB graph but converse of this statement is not true.

Theorem 6.1 : The geometric dual of LB graph if no loop present inside of other loops, is a L* graph.

Proof: Let G be a LB graph with n vertices and m loops such that no loop present inside of other loops. The geometric dual of G say G* has m+1 vertices and m+n-1 edges. So G* has n-1 loops. G* has m pendent vertices and one vertex is of degree (2n + m – 2) ≥ 3. Thus graph G* is L* (m+1, n-1) graph.

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