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# A New Class of Analytic Functions with Fekete-Szego Inequality Using Subordination Method 

S.K. Gandhi ${ }^{1}$, Gurmeet Singh $^{2}$, Preeti kumawat ${ }^{3}$, G.S. Rathore ${ }^{4}$, Lokendra kumawat ${ }^{5}$<br>1,3,4,5 Department of Mathematics and Statistics, Mohanlal Sukhadia University, Udaipur(Raj) 313001 (India)<br>${ }^{2}$ Department of Mathematics<br>GSSDGS Khalsa College, Patiala, (India)<br>Email: gandhisk28@gmail.com ${ }^{1}$, meetgur111@gmail.com ${ }^{2}$ preeti.kumawat30@gmail.com ${ }^{3}$, ganshyamsrathore@yahoo.co.in ${ }^{4}$<br>lokendrakumawat@yahoo.co.on ${ }^{5}$


#### Abstract

In this Paper we have introduced a new class of analytic functions and its subclasses by using principle of subordination and obtained sharp upper bounds of the function


$f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n}$ belonging to these classes.
Keywords: Bounded functions, Inverse Starlike functions, Starlike functions, Univalent functions and extremal function.

## 1. INTRODUCTION

Let $\boldsymbol{\mathcal { A }}$ denotes the class of functions of the form

$$
\begin{equation*}
f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n} \tag{1.1}
\end{equation*}
$$

which are analytic in the unit disc $\mathbb{E}=\{\mathbf{z}:|\mathbf{z}|<1 \mid\}$. Let $\boldsymbol{S}$ be the class of functions of the form Eq. (1.1), which are analytic univalent in $\mathbb{E}$.

Bieber Bach [7] proved that $\left|\boldsymbol{a}_{2}\right| \leq \mathbf{2}$ for the functions $f(\mathbf{z}) \in \mathcal{S}$.
And Löwner[6] proved that $\left|\boldsymbol{a}_{3}\right| \leq \mathbf{3}$ for the functions $\boldsymbol{f}(\mathbf{z}) \in \boldsymbol{\mathcal { S }}$.
With the known estimates $\left|\boldsymbol{a}_{\mathbf{2}}\right| \leq 2$ and $\left|\boldsymbol{a}_{\mathbf{3}}\right| \leq \mathbf{3}$, it was natural to seek some relation between $\boldsymbol{a}_{\mathbf{3}}$ and $\boldsymbol{a}_{\mathbf{2}}{ }^{\mathbf{2}}$ for the class $\boldsymbol{S}$. Fekete and Szego [9] used Löwner's[6] method to prove
the following well known result for the class $\boldsymbol{S}$.

Let $\boldsymbol{f}(\mathbf{z}) \in \boldsymbol{\mathcal { S }}$, then

$$
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq\left\{\begin{array}{cl}
3-4 \mu & \text {, if } \mu \leq 0  \tag{1.2}\\
1+2 e^{\left(\frac{-2 \mu}{1-\mu}\right)} & \text {, if } 0 \leq \mu \leq 1 \\
4 \mu-3 & \text {, if } \mu \geq 1
\end{array}\right.
$$

The inequality Eq. (1.2) plays a very important role in determining estimates of higher coefficients for some subclasses $\boldsymbol{S}$ (Chhichra[10], Babalola[5]).

Let us define some subclasses of $\boldsymbol{S}$.
We denote the class of Univalent convex functions $\boldsymbol{h}(\mathbf{z})=\boldsymbol{z}+\sum_{\boldsymbol{n}=2}^{\infty} \boldsymbol{c}_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \in \mathcal{A}$ by $\mathcal{K}$, satisfying the condition

$$
\begin{equation*}
\boldsymbol{\operatorname { R e }} \frac{\left(\left(z \boldsymbol{h}^{\prime}(z)\right)\right.}{\boldsymbol{h}^{\prime}(z)}>0, \quad z \in \mathbb{E} . \tag{1.3}
\end{equation*}
$$

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A function $\boldsymbol{f}(\mathbf{z}) \in \boldsymbol{\mathcal { A }}$ is said to be close to convex if there exist $\boldsymbol{g}(\mathbf{z}) \in \mathbf{S}^{*}$ such that

$$
\begin{equation*}
\boldsymbol{\operatorname { R e }} \frac{\left(\left(z f^{\prime}(z)\right)\right.}{g(z)}>0, \quad z \in \mathbb{E} . \tag{1.4}
\end{equation*}
$$

The class of close to convex functions introduced by Kaplan[16], is denoted by C and it was shown by him that all close to convex functions are univalent.

$$
\begin{equation*}
\mathrm{S}^{*}(\mathrm{~A}, \mathrm{~B})=\left\{f(z) \in \mathcal{A} ; \frac{\left(\left(z f^{\prime}(z)\right)\right.}{g(z)}<\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}, \quad-1 \leq B \leq A \leq 1, z \in \mathbb{E}\right\} \tag{1.5}
\end{equation*}
$$

Where $S^{*}(A, B)$ is a subclass of $S^{*}$.
Fekete-Szegö problem for strongly alpha quasi-convex functions was studied by Abdel-Gawad[3]. For different functions in the class $S$, the upper bound of $\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right|$ has been investigated by many authors including Goel and Mehrok[12] and in recent times by Al-Shaqsi and Darus[4], Hayami and Owa[15], AlAbbadi and Darus[1].

Gurmeet Singh et al.[2] introduced the class of inverse Starlike functions for the functions $\mathrm{g}(\mathrm{z})=\mathrm{z}+\sum_{n=2}^{\infty} b_{n} z^{n} \in \boldsymbol{\mathcal { A }}$, satisfying the condition

$$
\begin{equation*}
\operatorname{Re}\left(\frac{z f(z)}{2 \int_{0}^{z} f(z) d z}\right)>0, z \in E \quad \text { i.e. } \frac{z f(z)}{2 \int_{0}^{z} f(z) d z}<\frac{1+z}{1-z} \tag{1.6}
\end{equation*}
$$

We introduce the class $\mathcal{A}$ of Univalent Starlike functions $\boldsymbol{g}(\mathbf{z})=\mathbf{z}+\sum_{n=2}^{\infty} \boldsymbol{b}_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \in \boldsymbol{\mathcal { A }}$, satisfying the condition

$$
\begin{equation*}
\left[\frac{z\{z f(z)\}^{\prime}}{2 f(z)}\right]<\frac{1+z}{1-z} \quad \alpha>0 \tag{1.7}
\end{equation*}
$$

The subclass of $\boldsymbol{\mathcal { A }}$ consisting of the functions $\boldsymbol{g}(\mathbf{z})=\boldsymbol{z}+\sum_{\boldsymbol{n}=\mathbf{2}}^{\infty} \boldsymbol{b}_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \in \mathcal{A}$, satisfying the condition

$$
\begin{equation*}
\left[\frac{z\{z f(z)\}^{\prime}}{2 f(z)}\right]<\frac{1+\mathrm{Az}}{1+\mathrm{Bz}} ; \quad-1 \leq B \leq A \leq 1 \tag{1.8}
\end{equation*}
$$

Here, Symbol $\prec$ stands for subordination, define as follows:

## Principle of subordination:

If $\boldsymbol{f}(\mathbf{z})$ and $\boldsymbol{F}(\mathbf{z})$ are two functions which are analytic in $\mathbb{E}$, then $\boldsymbol{f}(\mathbf{z})$ is called a subordinate to $\boldsymbol{F}(\mathbf{z})$ in $\mathbb{E}$, if there exists a function $\boldsymbol{w}(\mathbf{z})$ which is analytic in $\mathbb{E}$ satisfying the conditions

$$
\text { (1) } \boldsymbol{w}(\mathbf{0})=\mathbf{0} \quad \text { and } \quad(2)|\boldsymbol{w}(\boldsymbol{z})|<1
$$

such that $\boldsymbol{f}(\mathbf{z})=\boldsymbol{F}(\boldsymbol{w}(\mathbf{z}))$, where $\mathbf{z} \in \mathbb{E}$ and we write it as $\boldsymbol{f}(\mathbf{z})<\boldsymbol{F}(\mathbf{z})$.
We denote the class of analytic bounded functions of the form
$w(z)=\sum_{n=1}^{\infty} d_{n} z^{n}, w(0)=0,|w(z)|<1 \quad$ by $\mathcal{U}$.
Here, $\left|\boldsymbol{d}_{\mathbf{1}}\right| \leq 1,\left|\boldsymbol{d}_{\mathbf{2}}\right| \leq \mathbf{1}-\left|\boldsymbol{d}_{\mathbf{1}}\right|^{\mathbf{2}}$.

## 2. RESULT AND DISCUSSION:

THEOREM 1.1: If $\boldsymbol{f}(\mathbf{z}) \in \mathcal{A}$, then the result

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$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq\left\{\begin{array}{lr}
10-16 \mu & \text {, if } \mu \leq \frac{1}{2}  \tag{1.10}\\
2 & \text {, if } \frac{1}{2} \leq \mu \leq \frac{3}{4} \\
16 \mu-10 & \text {,if } \mu \geq \frac{3}{4}
\end{array}\right.
$$

is sharp.
Proof: By definition of $\boldsymbol{\mathcal { A }}$,we have

$$
\begin{equation*}
\left[\frac{z\{z f(z))^{\prime}}{2 f(z)}\right]<\frac{1+z}{1-z} \tag{1.13}
\end{equation*}
$$

On Expanding Eq. (1.13) we have

$$
\begin{equation*}
1+\frac{1}{2} \boldsymbol{a}_{2} z+\left(\boldsymbol{a}_{3}-\frac{1}{2} \boldsymbol{a}_{2}^{2}\right) z^{2}+---=1+2 c_{1} z+2\left(c_{2}+c_{1}^{2}\right) z^{2}+---- \tag{1.14}
\end{equation*}
$$

After identifying the terms in Eq. (1.14), we have

$$
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq\left|2 c_{2}+10 c_{1}^{2}-16 \mu c_{1}^{2}\right|
$$

This leads to

$$
\begin{equation*}
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq 2+[|10-16 \mu|-2]\left|\boldsymbol{c}_{1}\right|^{2} \tag{1.15}
\end{equation*}
$$

Case I: when, $\mu \leq \frac{5}{8}$, then Eq. (1.15) leads to

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq 2+(8-16 \mu)\left|\boldsymbol{c}_{1}\right|^{2} \tag{1.16}
\end{equation*}
$$

Subcase I(a): when , $\mu \leq \frac{1}{2}$, then Eq. (1.16) leads to

$$
\begin{equation*}
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq 10-16 \mu \tag{1.17}
\end{equation*}
$$

Subcase I(b): when, $\mu \geq \frac{1}{2}$, then Eq. (1.16) leads to

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq 2 \tag{1.18}
\end{equation*}
$$

Case II : when, $\mu \geq \frac{\mathbf{5}}{\mathbf{8}}$, then Eq. (1.15) leads to

$$
\begin{equation*}
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq 2+[16 \mu-(10+2)]\left|\boldsymbol{c}_{1}\right|^{2} \tag{1.19}
\end{equation*}
$$

Subcase II(a): when , $\mu \leq \frac{3}{4}$, then Eq. (1.16) leads to

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq 2 \tag{1.20}
\end{equation*}
$$

Subcase II(b) : when , $\mu \geq \frac{3}{4}$, then Eq. (1..16) leads to

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$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq 16 \mu-10 \tag{1.21}
\end{equation*}
$$

Combining subcase II(a) and subcase I(b), we get

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq 2 \quad, \text { if } \quad \frac{1}{2} \leq \mu \leq \frac{3}{4} \tag{1.22}
\end{equation*}
$$

This completes the theorem. Therefore the result is sharp.

Extremal function for the first and third inequality is given by

$$
\begin{equation*}
f_{1}(z)=\frac{z}{(1-z)^{4}} \tag{1.23}
\end{equation*}
$$

And Extremal function for the second inequality is given by

$$
\begin{equation*}
f_{2}(z)=\frac{z}{\left(1-z^{2}\right)^{2}} \tag{1.24}
\end{equation*}
$$

THEOREM 1.2: If $\boldsymbol{f}(\mathbf{z}) \in \mathcal{A}$, then the result

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq
$$

$$
\left\{\begin{array}{l}
(\mathrm{A}-\mathrm{B})(2 \mathrm{~A}-3 \mathrm{~B})-4 \mu(\mathrm{~A}-\mathrm{B})^{2} \quad, \text { if } \mu \leq \frac{2 A-3 B-1}{4(A-B)}  \tag{1.25}\\
(A-B) \quad, \quad \text { if } \frac{2 A-3 B-1}{4(A-B)} \leq \mu \leq \frac{2 A-3 B+1}{4(A-B)} \\
4 \mu(\mathrm{~A}-\mathrm{B})^{2}-(\mathrm{A}-\mathrm{B})(2 \mathrm{~A}-3 \mathrm{~B}) \quad, \text { if } \mu \leq \frac{2 A-3 B+1}{4(A-B)}
\end{array}\right.
$$

is sharp.
Proof: By definition of $\boldsymbol{\mathcal { A }}$ we have

$$
\begin{equation*}
\left[\frac{z\{z f(z))^{\prime}}{2 f(z)}\right]<\frac{1+\mathrm{Az}}{1+\mathrm{Bz}} \tag{1.28}
\end{equation*}
$$

On expanding Eq. (1.28) we have

$$
\begin{equation*}
1+\frac{1}{2} a_{2} z+\left(a_{3}-\frac{1}{2} a_{2}^{2}\right) z^{2}+---=1+(\mathrm{A}-\mathrm{B}) c_{1} z+(\mathrm{A}-\mathrm{B})\left(c_{2}-B c_{1}^{2}\right) z^{2}+-- \tag{1.29}
\end{equation*}
$$

After identifying the terms in Eq. (1.29), we have

$$
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq\left|(\mathrm{A}-\mathrm{B})\left(c_{2}-B c_{1}^{2}\right)+2(\mathrm{~A}-\mathrm{B})^{2} c_{1}^{2}-4 \mu(\mathrm{~A}-\mathrm{B})^{2} c_{1}^{2}\right|
$$

This leads to

$$
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq(\mathrm{A}-\mathrm{B})+\left\{\left|2(\mathrm{~A}-\mathrm{B})^{2}-B(A-B)-4 \mu(\mathrm{~A}-\mathrm{B})^{2}\right|-(\mathrm{A}-\mathrm{B})\right\}\left|\boldsymbol{c}_{\boldsymbol{1}}\right|^{2}
$$

Case I : when, $\mu \leq \frac{2 A-3 B}{4(A-B)}$, then Eq. (1.30) leads to

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$$
\begin{equation*}
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq(\mathrm{A}-\mathrm{B})+\left\{(\mathrm{A}-B)(2 A-3 B-1)-4 \mu(\mathrm{~A}-\mathrm{B})^{2}\right\}\left|\boldsymbol{c}_{\mathbf{1}}\right|^{2} \tag{1.31}
\end{equation*}
$$

Subcase I(a): when, $\mu \leq \frac{2 A-3 B-1}{4(A-B)}$, then Eq. (1.31) leads to

$$
\begin{equation*}
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq \quad\left\{(\mathrm{A}-\mathrm{B})(2 \mathrm{~A}-3 \mathrm{~B})-4 \mu(\mathrm{~A}-\mathrm{B})^{2}\right. \tag{1.32}
\end{equation*}
$$

Subcase I(b) : when , $\mu \geq \frac{2 A-3 B-1}{4(A-B)}$, then Eq. (1.31) leads to

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq(\mathrm{A}-\mathrm{B}) \tag{1.33}
\end{equation*}
$$

Case II : when, $\mu \geq \frac{2 A-3 B}{4(A-B)}$, then Eq. (1.30) leads to

$$
\begin{equation*}
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq(\mathrm{A}-\mathrm{B})+\left\{4 \mu(\mathrm{~A}-\mathrm{B})^{2}-(\mathrm{A}-\mathrm{B})(2 \mathrm{~A}-3 \mathrm{~B}+1)\right\}\left|\boldsymbol{c}_{1}\right|^{2} \tag{1.34}
\end{equation*}
$$

Subcase II(a): when, $\mu \leq \frac{2 A-3 B+1}{4(A-B)}$, then Eq. (1.34) leads to

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right|_{\leq}(\mathrm{A}-\mathrm{B}) \tag{1.35}
\end{equation*}
$$

Subcase II(B): when, $\mu \geq \frac{2 \boldsymbol{A - 3 B + 1}}{4(\boldsymbol{A}-\boldsymbol{B})}$, then Eq. (1.34) leads to

$$
\begin{equation*}
\left|\boldsymbol{a}_{3}-\mu \boldsymbol{a}_{2}^{2}\right| \leq \quad\left\{4 \mu(\mathrm{~A}-\mathrm{B})^{2}-(A-B)(2 A-3 B)\right. \tag{1.36}
\end{equation*}
$$

Combining subcase $\mathrm{II}(\mathrm{a})$ and subcase $\mathrm{I}(\mathrm{b})$, we get

$$
\begin{equation*}
\left|a_{3}-\mu a_{2}^{2}\right| \leq(\mathrm{A}-\mathrm{B}) \quad, \quad \text { if } \frac{2 A-3 B-1}{4(A-B)} \leq \mu \leq \frac{2 A-3 B+1}{4(A-B)} \tag{1.37}
\end{equation*}
$$

This completes the theorem. Therefore the result is sharp.
Extremal function for the first and third inequality is given by

$$
\begin{equation*}
f_{1}(z)=\mathrm{z}(1+\mathrm{B} z)^{\frac{2(A-B)}{B}} \tag{1.38}
\end{equation*}
$$

Extremal function for the second inequality is given by

$$
\begin{equation*}
f_{2}(z)=\frac{z}{\left(1-z^{2}\right)^{A-B}} \tag{1.39}
\end{equation*}
$$

## 3. CONCLUSION:

If we take $\mathrm{A}=1$ and $\mathrm{B}=-1(-\mathbf{1} \leq \boldsymbol{B} \leq \boldsymbol{A} \leq \mathbf{1})$ in the result of theorem 1.2 , we get the result of theorem 1.1, therefore our result for the theorem 1.2 reduces to the result of the theorem 1.1. And the results are sharp and also if we put $\mathrm{A}=1$ and B $=-1$ in the extremal function of theorem 1.2 , we get the extremal function of theorem 1.1. Hence theorem 1.2 is the generalization of theorem 1.1.

## REFERENCES

[1] Al-Abbadi, M. H. and M.Darus. "FeketeSezegö theorem for a certain class of analytic functions". Sanis Malaysiana, volume 40, no. 4, pages 385-389,2011.
[2] Gurmeet Singh, M. S. Saroa, and. B. S. Mehrok. "Fekete-szegö inequality for a New class of analytic functions". Elsevier, Proc. of International conference on

Information and Mathematical Sciences, pages 90-93,2013.
[3] H. R Abdel-Gawad, and D.K. Thomas. "The Fekete-Szegö problem for strongly close-toconvex functions" .Proceedings of the Amercian Mathematical Society, volume 114, no.2, pages 345-349,1992.
[4] K. Al-Shaqsi and M. Darus. "On FeketeSzegö problems for certain subclass of analytic Functions".Applied Mathematical Sciences, volume 2 no. 9-12,pages.431441,2008.
[5] Kunle Oladeji Babalola "The fifth and Sixth coefficient of $\alpha$ - close- to -convex Function". Kragujevac J. Math. 32, pages 5-12,2009.
[6] K. Löwner. "Uber monotone Matrixfunktionen". Math. Z 38, pages 177216, 1934.
[7] L. Bieberbach. " Uber einige Extremal Probleme im Gebiete der Konformen Abbildung". Math. 77,pages 153-172,1916
[8] L. Bieberbach. " Uberdie Koeffizientem derjenigem Potenzreihen, welche eine Schlithe Abbildung des Einheitskrises Vermittelen. Preuss". AKad. Wiss
Sitzungsb. pages 940-955, 1916.
[9] M. Fekete and G.Szego. "Eine Bemerkung Uber ungerade Schlichte Funktionen". J. London Math. Soc. 8 , pages 85-89,1933.
[10] P. N. Chhichra. "New Subclasses of the class of close to convex functions". Procedure of American Mathematical Society, 62 ,pages 37-43, 1977.
[11] R. M. Goel, and B. S. Mehrok. "A Subclass of Univalent functions". Houston Journal of Mathematics 8, pages 343-357, 1982.
[12] R.M. Goel, and B. S. Mehrok. "A coefficient inequality for certain classes of analytic function" Tamkang Journal of Mathematics 22, pages 153-163,1990.
[13] S. R. Keogh, and E. R. Merkes. "A coefficient inequality for certain classes of analytic functions".Proceedings of the American Mathematical Society. volume 20, pages 812,1969.
[14] S. R. Keogh,and E. R. Merkes."A coefficient inequality for certain classes of analytic functions". Procedure of American Mathematical Society, 20, pages 8-12,1989.
[15] T. Hayami,S.Owa and H. M.Srivastava. "Coefficient inequalities for certain classes of analytic and univalent functions". J. Ineq. Pure and Appl. Math. 8(4), 2007.
[16]W. Kaplan. "Close-to-convex schlicht functions". Michigan Mathematical Journal 1,pages 169-185, 1952.

Corresponding Author: Preeti Kumawat Department of Mathematics and Statistics (Research scholar)
Mohanlal sukhadia university Udaipur [Raj.], INDIA
preeti.kumawat30@gmail.com 9530080280

