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Analysis of an M/M/c queue with Single Working Vacation and Impatient Customers

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Abstract- We analyse the impatient customers in an infinite capacity M/M/c queue with single working vacation, where the server provides service to the customers at a reduced rate rather than stopping the service completely during his vacation period. A customer waiting in a queue becomes impatient if he finds the server in working vaction period. We constructed the model as a quasi-birth–death process and form the steady state equations. We derived the probability generating function for the number of customers present when the server is both, in a service period as well as in a working vacation period and obtained various performance measures. Stochastic decomposition structures of the queue length and waiting time are derived.

Keywords: M/M/c queue, Working vacation, Impatient customers, Stochastic decomposition, Generating function

1. INTRODUCTION

Impatient customers in queuing models occur in several life scenarios such as those involving impatient telephone switchboard customers, hospital emergency rooms handling critical patients and inventory systems that store perishable goods. Queueing systems with impatient customers have been analysed by many authors such as Benjaafar et al. (2010) and Bonald et al. (2001), and considered the impatience behavior by various directions, due to their potential applications in call centers, communication networks, production-inventory systems and several other areas. The first to investigate the impatient phenomenon in queuing models appears to be be Palm's (1953) pioneering work, by considering the infinite buffer M/M/c queue where each arriving customer remains in the queue until his waiting time does not exceed the impatient time which is exponentially distributed. Daley (1965) analyzed the impatient phenomenon in *GI/G/*1 queuing system in which the customers may leave the system if their waiting time is too long before starting or completing their service. Delay obtained an integral equation for the limiting distribution function and analyzed solution for the deterministic and distributed impatience. Takacs (1974) further analyzed the M/G/1 queuing system in which customers sojourn time has a static threshold and obtained the actual and virtual limiting waiting time distributions. And these results are generalized in different direction by several authors like Baccelli et al. (1984), Boxma et al. (1994), Van Houdt et al. (2003) and Yue et al. (2009).

In the above mentioned literature, the cause of impatience was either a long wait already experienced by a customer upon arrival at a queue, or a long wait anticipated by a customer upon arrival. However, Altman and Yechiali (2006, 2008) studied queuing models with impatient customers where the cause of impatience becomes the server's vacation and unavailability of server upon arrival. Hence, the cause of the impatience is the unavailability of the server. The M/M/1, M/G/1 and M/M/c queues were analyzed in Altman and Yechiali (2006), whereas $M/M/\infty$ queue was studied in Altman and Yechiali (2008). Yechiali (2007) investigated the queueing model with system disasters where the customers turned to be impatient only when the system is down. This work was broaden and enhanced by Economou and Kapodistria (2010) who studied synchronized abandonments in queuing models. Perel and Yechiali (2010) analyzed M/M/c queuing system with impatient customers operating in a 2-phase (fast and slow) Markovian random environment. Customers became impatient because of the slow service rate when the system works in a slow phase. Yue et al. (2016) and Kawanishi (2008) investigated the impatience behaviour of a finite capacity multi-server queuing system. Yue et al. (2014) derived the closed-form solution of different performance measures in an M/M/1 queue with impatient customers and variant of multiple vacation policy.

In the above mentioned study, we have assumed that the server halts service during the vacation. However, there are lot of examples where the server does not completely stop serving the customers during the vacation, rather it will render service at a lower rate to the queue. Servi and Finn (2002) were the first to introduce this kind of vacation policy, called working vacation policy and studied an M/M/1/WV queuing model where service times during a non-vacation period, the service times during a working vacation, and the vacation times are all assumed to be exponentially distributed with different rates. Kim et al. (2003), Wu and Takagi (2006) generalized the work of Servi and Finn (2002) to an M/G/1 queue with working vacation. Baba (2005) extended this study by using the matrix-geometric method to a GI/M/1 queue with working vacation. Tian et al. (2008) investigated the M/M/1 queue with single working vacation. Banik et al. (2007) studied the GI/M/1/N queue with multiple working vacations and computed a series of numerical results. Jain and Upadhyaya (2011) analyzed a finite-buffer multi-server unreliable Markovian queue with synchronous working vacation policy. Banik (2010) studied the *GI/M/*1*/N* and *GI/M/*1*/1* queuing models for single working vacation. Recently, Selvaraju and Goswami (2013) analysed the M/M/1 queue with single and multiple working vacation and impatient customers. They computed closed form solution and various performance measures with stochastic decomposition for both the working vacation policies.

The model is presented as follows. In section 2, we provide the description of the model. We formulate the model as a quasi-birth-death process and explicit expressions of the stationary probabilities are derived. In section 3, stochastic decomposition properties are verified.

2. MODEL DESCRIPTION

We consider an infinite capacity M/M/1 queuing model with working vacation, where the server provides service to the customers at a reduced rate rather than stopping the service completely during his vacation period. The customers arrive according to a Poisson process with parameter λ. The server serves the customers at an exponential rate μ_b during a non-vacation period, where we consider the stability condition that $\rho = \frac{\lambda}{c\mu_b} < 1$. The service discipline is first come first served (FCFS). The server begins a working vacation as soon as the system becomes unoccupied. The arriving customers during working vacation are served at a rate lower than the regular service rate. The service times during working vacation and vacation times are also assumed to be exponentially distributed with rates μ_v and θ, respectively.

A customer waiting in a queue becomes impatient if he finds the server in working vacation period i.e. if he finds the server serving at rate μ_v, he activates an exponentially distributed impatient timer T with parameter ξ. The customer exits the queue and never
returns if its service has not been completed before the timer T expires. Thus to conclude that only those customers whose arrival occurs during a WV of the server, are impatient. This type of impatient policy is different from that of the impatient policy studied in Altman and Yechiali (2006) in which all arriving customers become impatient during the vacation period, since a pure vacation policy is considered. The interarrival times, service times, vacation duration times and impatient time are all taken to be mutually independent.

Let $\{N(t), t \geq 0\}$ be the number of customers in the system at time t and J (t) be the state of system at time t, where J(t) is defined as follows:

$$J(t) = \begin{cases} 1 & \text{when the servers are a non-vacation period at time t ,} \\ 0 & \text{when the servers are in WV period at time t.} \end{cases}$$

Then $\Delta = \{(N(t), J(t)), t \geq 0\}$ is a two dimensional continuous time discrete state Markov chain with state space $S = \{\{(0,0)\} \bigcup \{(i,j)\}, i = 1,2,..., j = 0,1\}$

2.1 The Stationary distribution

Let us define the stationary probabilities for a Markov chain Δ as

$$P_{ij} = P\{N(t) = i, J(t) = j\}, \ i = 0,1,2,.., j = 0,1$$

Then, the stationary equations are.

$$(\lambda + c\theta)P_{00} = (\mu_v + \xi)P_{1,0} + \mu_b P_{11},\ if\ n = 0 \quad (1)$$

$$[\lambda + \theta + n(\mu_v + \xi)]P_{n,0} = \lambda P_{n-1,0} + (n+1)(\mu_v + \xi)P_{n+1,0},\ if\ 1 \le n \le c-1 \quad (2)$$

$$[\lambda + \theta + c\mu_v + n\xi]P_{n,0} = \lambda P_{n-1,0} + [c\mu_v + (n+1)\xi]P_{n+1,0},\ if\ n \ge c \quad (3)$$

$$\lambda P_{0,1} = \theta p_{0,0},\ if\ n = 0 \quad (4)$$

$$(\lambda + n\mu_b)P_{11} = \lambda P_{n-1,1} + (n+1)\mu_b P_{n+1,1} + \theta P_{n,0}\ if\ 1 \le n \le c-1 \quad (5)$$

$$(\lambda + c\mu_b)P_{n,1} = \lambda P_{n-1,1} + c\mu_b P_{n+1,1} + \theta P_{n,0}\ if\ n \ge c \quad (6)$$

Now define the (partial) probability generating functions (pgf), for $0 < z < 1$

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0}, \quad P_1(z) = \sum_{n=1}^{\infty} z^n P_{n,1},$$

with $P_0(1) + P_1(1) = 1$ and $P_0'(z) = \sum_{n=1}^{\infty} n z^{n-1} P_{n,0}$,

Multiplying (2) and (3) with z^n and summing over n we get the following differential equation after using (1) and rearrangement of terms.

$$\xi z(1-z)P_0'(z) - [(1-z)(\lambda z - c\mu_v) + z\theta]P_0(z) + [\theta(1-c)z - c\mu_v(1-z)]P_{0,0} + z\mu_b P_{11} - \mu_v(1-z)\sum_{n=1}^{c}(n-c)P_{n,0}z^n = 0 \quad (7)$$

or

$$P_0'(z) - \left[\frac{\lambda z - c\mu_v}{z\xi} + \frac{\theta}{\xi(1-z)}\right]P_0 z + \left[\frac{\theta(1-c)}{\xi(1-z)} - \frac{c\mu_v}{z\xi)}\right]P_{0,0} + \frac{\mu_b}{\xi(1-z)}P_{1,1} - \frac{\mu_v}{\xi z}\sum_{n=1}^{c}(n-c)z^n P_{n,0} = 0 \quad (8)$$

In order to solve the above differential equation we multiply it both sides by

$$\text{I.F} = e^{-\int\left[\frac{\lambda z - c\mu_v}{\xi z} + \frac{\theta}{\xi(1-z)}\right]dz} = e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}} (1-z)^{\frac{\theta}{\xi}}$$, we get

$$\frac{d}{dz}\left[e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}} (1-z)^{\frac{\theta}{\xi}} P_0(z)\right] = \left\{-\left(\frac{\theta(1-c)}{\xi(1-z)} - \frac{c\mu_v}{z\xi}\right)P_{0,0} - \frac{\mu_b}{\xi(1-z)}P_{1,1} + \frac{\mu_v}{\xi z}\sum_{n=1}^{c}(n-c)z^n P_{n,0}\right\} e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}} (1-z)^{\frac{\theta}{\xi}}$$

Integrating from 0 to z, we get

$$P_0(z)=e^{\frac{\lambda z}{\xi}}z^{-\frac{c\mu_v}{\xi}}(1-z)^{-\frac{\theta}{\xi}}\{\frac{c\mu_v}{\xi}P_{0,0}\int_0^z e^{-\frac{\lambda z}{\xi}}z^{\frac{c\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}dz$$

$$-\left[\frac{\theta(1-c)}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right]\int_0^z e^{-\frac{\lambda z}{\xi}}z^{\frac{c\mu_v}{\xi}}(1-z)^{\frac{\theta}{\xi}-1}dz+\frac{\mu_v}{\xi}\int_0^z Q_1(z)e^{-\frac{\lambda z}{\xi}}z^{\frac{c\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}dz\} \quad (9)$$

where $$Q_1(z)=\sum_{n=1}^{c}(n-c)z^nP_{n,0} \quad (10)$$

$$P_0(z)=z^{-\frac{c\mu_v}{\xi}}(1-z)^{-\frac{\theta}{\xi}}\left[\frac{c\mu_v}{\xi}P_{0,0}A(z)-\left(\frac{(1-c)\theta}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right)B(z)+\frac{\mu_v}{\xi}C(z)\right] \quad (11)$$

where

$$A(z)=\int_0^z e^{\frac{\lambda}{\xi}(z-X)}X^{\frac{c\mu_v}{\xi}-1}(1-X)^{\frac{\theta}{\xi}}dX \quad (12)$$

$$B(z)=\int_0^z e^{\frac{\lambda}{\xi}(z-X)}X^{\frac{c\mu_v}{\xi}}(1-X)^{\frac{\theta}{\xi}-1}dX \quad (13)$$

$$C(z)=\int_0^z Q_1(X)e^{\frac{\lambda}{\xi}(z-X)}X^{\frac{c\mu_v}{\xi}-1}(1-X)^{\frac{\theta}{\xi}}dX \quad (14)$$

Now, determine $P_0(z)$ for $\lim_{z\to1}$ gives

$$\lim_{z\to1}P_0(z)=P_0(1)=\left[\frac{c\mu_v}{\xi}P_{0,0}A(1)-\left(\frac{\theta(1-c)}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right)B(1)+\frac{\mu_v}{\xi}c(1)\right]\lim_{z\to1}(1-z)^{\frac{-\theta}{\xi}}$$

Since $0\le P_0(1)=\sum_{n=0}^{\infty}P_{n,0}\le1$ and $\lim_{z\to1}(1-z)^{-\frac{\theta}{\xi}}=\infty$, so we must have the term

$$\frac{c\mu_v}{\xi}P_{0,0}A(1)-\left(\frac{\theta(1-c)}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right)B(1)+\frac{\mu_v}{\xi}C(1)=0$$

or

$$P_{1,1}=\frac{\mu_v}{\mu_b}\frac{C(1)}{B(1)}+\left[\frac{\mu_v}{\mu_b}\frac{A(1)}{B(1)}-\frac{\theta(1-c)}{\mu_b}\right]P_{0,0} \quad (15)$$

Using (14) in (10), we get

$$P_0(z)=\left\{\frac{c\mu_v}{\xi}\left[A(z)-\frac{A(1)}{B(1)}B(z)\right]P_{0,0}-\frac{\mu_v}{\xi}\left[\frac{C(1)}{B(1)}B(z)-C(z)\right]\right\}z^{\frac{c\mu_v}{\xi}}(1-z)^{-\frac{\theta}{\xi}} \tag{16}$$

Multiplying (5) and (6) by z^n and summing over n, we get after using (4)

$$(1-z)(\lambda z-c\mu_b)P_1(z)=z\theta P_0(z)-z(1-z)\theta P_{0,0}-\mu_b zP_{1,1}+\mu_b(1-z)\sum_{n=1}^{c}(n-c)z^nP_{n,1} \tag{17}$$

Using (13) and (15) , we get

$$P_1(z)=\left\{\frac{z\theta}{c\mu_b}P_o(z)-z\frac{\mu_v}{c\mu_b}\frac{C(1)}{B(1)}-z\left[\frac{\mu_v}{\mu_b}\frac{A(1)}{B(1)}-\frac{\theta(z-c)}{c\mu_b}\right]P_{0,0}+\frac{(1-z)Q(z)}{c}\right\}(z-1)^{-1}(1-\rho z)^{-1} \tag{18}$$

where

$$Q(z)=\sum_{n=1}^{c}(n-c)P_{n.1}z^n \tag{19}$$

Theorem 2.1.1 *If* $\rho<1$ *and* $0<\xi<\mu_v$ *, then the probability* $P_{0,0}$ *is given by*

$$P_{0,0}=\frac{(\theta+\xi)\left[(c\mu_b-\lambda)-\mu_bQ(1)\right]-\mu_v\theta Q_1(1)-\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}\left[\theta(\lambda-c\mu_v)+(\theta+\xi)(c\mu_b-\lambda)\right]}{c\mu_v\theta+(\theta+\xi)(2-c)\theta+\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}\left[\theta(\lambda-c\mu_v)+(\theta+\xi)(\mu_bc-\lambda)\right]} \tag{20}$$

Proof. Applying L'Hosipital rule to (10), we get

$$\lim_{z\to1}P_0(z)=\lim_{z\to1}\frac{\left\{\frac{c\mu_v}{\xi}P_{0,0}(1-z)-\left[\frac{\theta(1-c)}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right]z+\frac{\mu_v}{\xi}Q_1(z)(1-z)\right\}}{c\frac{\mu_v}{\xi}(1-z)-\frac{\theta}{\xi}z} \tag{21}$$

which gives

$$\theta P_0(1)=\theta(1-c)P_{0,0}+\mu_bP_{1,1} \tag{22}$$

Similarly from (17), we get

$$\lim_{z\to1}P_1(z)=\lim_{z\to1}\left[\frac{\theta P_0(z)+z\theta P_0'(z)-\theta P_{0,0}[-z+(1-z)]-\mu_bP_{1,1}}{\lambda(1-z)-(\lambda z-c\mu_b)}\right]-\frac{\mu_b}{\lambda-c\mu_b}Q(1)$$

Applying (21),we get

$$P_1(1) = \frac{\theta P_0(1) + \theta P_0'(1) + \theta P_{0,0} - \mu_b P_{1,1}}{c\mu_b - \lambda} - \frac{\mu_b}{\lambda - c\mu_b} Q(1)$$
$$= \frac{\theta(2-c)P_{0,0} + \theta P_0'(1) + \mu_b Q(1)}{c\mu_b - \lambda}$$

which implies that

$$P_0'(1) = \left(\frac{c\mu_b - \lambda}{\theta}\right) P_1(1) - (2-c)P_{0,0} - \frac{\mu_b}{\theta} Q(1) \tag{23}$$

From (7), we get

$$P_0'(z) = \frac{[(1-z)(\lambda z - c\mu_v) + z\theta]P_0(z) - [\theta(1-c)z - c\mu_v(1-z)]P_{0,0} - z\mu_b P_{1,1}}{\xi z(1-z)} + \frac{\mu_v}{\xi z} Q_1(z)$$

Applying L-hospitals rule, we get

$$P_0'(1) = \lim_{z\to 1}\left[\frac{(\lambda(1-2z)+c\mu_v+\theta)P_0(z)+P_0'(z)[(1-z)\lambda z-c\mu_v+z\theta]-[\theta(1-c)+c\mu_v]P_{0,0}-\mu_b P_{1,1}}{\xi(1-\lambda z)}\right] + \frac{\mu_v}{\xi} Q_1(1)$$

Therefore using (21), we get

$$P_0'(1) = \frac{(\lambda - c\mu_v)P_0(1) + c\mu_v P_{0,0} + \mu_v Q_1(1)}{\theta + \xi} \tag{24}$$

From (22) and (23)

$$\left(\frac{c\mu_b - \lambda}{\theta}\right) P_1(1) - (2-c)P_{0,0} - \frac{\mu_b}{\theta} Q(1) = \frac{(\lambda - c\mu_v)P_0(1) + c\mu_v P_{0,0} + \mu_v Q_1(1)}{\theta + \xi}$$

Which simplifies to

$$[\theta(\lambda - c\mu_v) + (\theta+\xi)(\mu_b c - \lambda)][(1-c)P_{0,0} + \frac{\mu_b}{\theta} P_{1,1}] + c\theta\mu_v P_{0,0} + (\theta+\xi)(2-c)\theta P_{0,0}$$
$$= (\theta+\xi)[(\mu_b c - \lambda) - \mu_b Q(1)] - \mu_v \theta Q_1(1)$$

Hence putting $P_{1,1}$ in terms of $P_{0,0}$, we get the expression for $P_{0,0}$ as

$$P_{0,0} = \frac{(\theta+\xi)[(\mu_b c - \lambda) - \mu_b Q(1)] - \mu_v \theta Q_1(1) - \frac{\mu_v}{\theta}\frac{C(1)}{B(1)}[\theta(\lambda - c\mu_v) + (\theta+\xi)(\mu_b c - \lambda)]}{c\mu_v\theta + (\theta+\xi)(2-c)\theta + \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}[\theta(\lambda - c\mu_v) + (\theta+\xi)(\mu_b c - \lambda)]} \tag{25}$$

2.2. Performance measures

From (21) the equilibrium probability that the system is in working vacation is

$$P_0(1) = \frac{\mu_v}{\theta}\frac{C(1)}{B(1)} + \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)} P_{0,0} \tag{26}$$

and the probability that the system is in non vacation period is

$$P_1(1) = 1 - \frac{\mu_v}{\theta}\frac{C(1)}{B(1)} - \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0} \tag{27}$$

The mean number of customers when the system is on WV vacation period is

$$E(N_0) = P_0(1) = \left(\frac{c\mu_b - \lambda}{\theta}\right)\left[1 - \frac{\mu_v}{\theta}\frac{C(1)}{B(1)} - \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0}\right] - (2-c)P_{0,0} - \frac{\mu_b}{\theta}Q(1) \tag{28}$$

and the expected number of customers when the server is on non vacation period is

$$E(N_1) = P_1'(1) = \frac{\theta}{\mu_b}E(N_0) = \left(\frac{c\mu_b - \lambda}{\mu_b}\right)\left[1 - \frac{\mu_v}{\theta}\frac{C(1)}{B(1)} - \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0}\right] - \frac{\theta(2-c)}{\mu_b}P_{0,0} - Q(1) \tag{29}$$

Hence the steady state mean number of customers in the system is

$$E(N) = E(N_0) + E(N_1)$$

$$= (c\mu_b - \lambda)\left[\frac{1}{\theta} + \frac{1}{\mu_b}\right]\left[1 - \frac{\mu_v}{\theta}\frac{C(1)}{B(1)} - \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0}\right] - (2-c)P_{0,0}\left(1 + \frac{\theta}{\mu_b}\right) - \left(1 + \frac{\mu_b}{\theta}\right)Q(1) \tag{30}$$

Using little formula,the mean waiting time in the system can be obtained as

$$E(W) = \frac{E(N)}{\lambda}$$

Another importance measure

$$E(W_{n,1}) = \frac{n+1}{\mu_b}, n = 0,1,2,... \tag{31}$$

We derive $E(W_{n,0})$ by using the method of Altman and Yechiali (2006).

For $j = 0$ and $n \geq 1$,

$$\begin{aligned} E(W_{n,0}) &= \frac{\theta}{\theta + \lambda + (n+1)(\mu_v + \xi)}\left[\frac{1}{\theta + \lambda + (n+1)(\mu_v + \xi)} + E(W_{n,1})\right] \\ &+ \frac{\lambda}{\theta + \lambda + (n+1)(\mu_v + \xi)}\left[\frac{1}{\theta + \lambda + (n+1)(\mu_v + \xi)} + E(W_{n,0})\right] \\ &+ \frac{n\xi}{\theta + \lambda + (n+1)(\mu_v + \xi)}\left[\frac{1}{\theta + \lambda + (n+1)(\mu_v + \xi)} + E(W_{n-1,0})\right] \\ &+ \frac{n\mu_v}{\theta + \lambda + (n+1)(\mu_v + \xi)}\left[\frac{1}{\theta + \lambda + (n+1)(\mu_v + \xi)} + E(W_{n-1,0})\right] \end{aligned}$$

The second term above follows from the fact that a new arrival does not change the waiting time of a customer present in the system, while the third term takes into consideration that only n customers can abandon the system as our customer is not impatient.

$$E(W_{n,0}) = \frac{1}{\theta+(n+1)(\mu_v+\xi)}\{\frac{\theta+\lambda+n(\mu_v+\xi)}{\theta+\lambda+(n+1)(\mu_v+\xi)}+\frac{(n+1)\theta}{\mu_b}+n(\mu_v+\xi)E(W_{n-1},0)\} \quad (32)$$

For j=0 and n=0,

$$E(W_{0,0}) = \frac{\theta}{\theta+\lambda+\mu_v+\xi}\left[\frac{1}{\theta+\lambda+\mu_v+\xi}+\frac{1}{\mu_b}\right]+\frac{\lambda}{\theta+\lambda+\mu_v+\xi}\left[\frac{1}{\theta+\lambda+\mu_v+\xi}+E(W_{0,0})\right]$$

which can be simplied to

$$E(W_{0,0}) = \frac{1}{\theta+\mu_v+\xi}\left[\frac{\theta+\lambda}{\theta+\lambda+\mu_v+\xi}+\frac{\theta}{\mu_b}\right] \quad (33)$$

using (33) and iterating (32), we obtain for $n \geq 0$

$$E(W_{n,0}) = \frac{1}{\theta+(n+1)(\mu_v+\xi)}\left\{\frac{\theta+\lambda+n(\mu_v+\xi)}{\theta+\lambda+(n+1)(\mu_v+\xi)}+\frac{(n+1)\theta}{\mu_b}\right.$$
$$\left.+\sum_{k=1}^{n}\left[\frac{\theta+\lambda+(k-1)(\mu_v+\xi)}{\theta+\lambda+k(\mu_v+\xi)}+\frac{k\theta}{\mu_b}\prod_{i=k}^{n}\frac{i(\mu_v+\xi)}{\theta+i(\mu_v+\xi)}\right]\right\}$$

Finally we get the mean waiting time of customers served by the system as

$$E(W_{served}) = \sum_{n=0}^{\infty}P_{n,0}E(W_{n,0})+\sum_{n=0}^{\infty}P_{n,1}E(W_{n,1})$$

which after using (30) becomes

$$E(W_{served}) = \sum_{n=0}^{\infty}P_{n,0}E(W_{n,0})+\frac{E(N_1)+P_1(1)}{\mu_b} \quad (34)$$

3. STOCHASTIC DECOMPOSITIONS RESULTS

Theorem 3.1. 2 *For $\rho < 1$,the stationary queue length N can be decomposed into a sum of two independent random variables as $N = N_c + N_d$ where N_c is the queue length of a classical M/M/c queue with vacation and N_c is the additional queue length due to effect of SWV with its PGF*

$$N_d(z) = \frac{P_{0,0}}{(c\mu_b-\lambda)(1-z)}\{\left[\frac{(c\mu_b-z\lambda)(1-z)-z\theta}{P_{00}}\right]P_0(z)$$
$$+z\left[\frac{\mu_v C(1)}{P_{00}B(1)}+(c-z)\theta+\frac{c\mu_v A(1)}{B(1)}\right]-\frac{\mu_b(1-z)}{P_{00}}Q(z)\} \quad (35)$$

Proof.

$$N(z) = P_0(z) + P_1(z)$$

$$= \left[1 + \frac{z\theta}{(1-z)(\lambda z - c\mu_b)}\right] P_0(z) - z(1-z)\theta P_{0,0}$$

$$- \frac{z\left(\frac{\mu_v C(1)}{B(1)} + \left[C\mu_v \frac{A(1)}{B(1)} - \frac{\theta(1-c)}{\mu_b}\right] P_{0,0}\right) - \mu_b (1-z)\theta(z)}{(1-z)(\lambda z - c\mu_b)}$$

$$= \left(\frac{c\mu_b - \lambda}{c\mu_b - \lambda z}\right) \left\{ \left[\frac{c\mu_b - \lambda z}{c\mu_b - \lambda} - \frac{\theta z}{(c\mu_b - \lambda)(1-z)}\right] P_0(z) \right.$$

$$\left. + \frac{z}{(1-z)(c\mu_b - \lambda)} \left[\frac{\mu_v c(1)}{B(1)} - \left((z-c)\theta - c\mu_v \frac{A(1)}{B(1)}\right) P_{0,0}\right] - \frac{\mu_b}{c\mu_b - \lambda} Q(z) \right\}$$

$$or\ \ N(z) = \frac{1-\rho}{1-\rho z} \times N_d(z)$$

$P_0(z)$ and $P_1(z)$ are positive, as they are PGFs and so $P_0(z) + P_1(z) > 0$ and for $0 < z < 1$ and $\rho < 1$, $\left(\frac{1-\rho}{1+\rho z}\right) > 0$. Therefore $N_d(z)$ is positive. Also, for z=1, $N_d(1) = 1$. Hence $N_d(z)$ is a PGF.

Theorem 3.2. 3 *If* $\rho < 1$, *the stationary waiting time can be decomposed into a sum of two independent random variable as* $W = W_c + W_d$, Where W_c is the waiting time of a customer corresponding to classical M/M/c queue and has exponential distribution with parameter $\mu_b(1-\rho)$ and W_d is the additional delay due to the effect of SWV with its Laplace stieltjes transform (LST)

$$W_d^*(s) = \frac{P_{0,0}}{(c\mu_b - \lambda)s} \left\{ \left[\frac{c\mu_b + s - \lambda)s - \theta(\lambda - s)}{P_{0,0}}\right] P_0\left(1 - \frac{s}{\lambda}\right) \right.$$

$$\left. + (\lambda - s)\left[\frac{\mu_v}{P_{0,0}} \frac{C(1)}{B(1)} + \left(c - 1 + \frac{s}{\lambda}\right)\theta + \frac{c\mu_v A(1)}{B(1)}\right] - \frac{\mu_b}{P_{0,0}} sQ\left(1 - \frac{s}{\lambda}\right) \right\} \tag{36}$$

Proof. From the distributional form of little's law, in Keilson and Servi (1988), we have the relation $N(z)$ = $W_d^*(\lambda(1-z))$. Let s= $\lambda(1-z)$ which gives z= $(1-\frac{s}{\lambda})$ and $1 - z = \frac{s}{\lambda}$.Putting these values in (35), we get the desired expression.

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