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The Mersenne Meet Matrices with A – Sets on Exponential Divisor Closed Sets

Dr. N. Elumalai[1] And R. Kalpana[2]
1. *Associate Professor of Mathematics, A.V.C.College (Autonomous) ,Mannampandal – 609 305, Mayiladuthurai, India. E-mail : nelumalai@rediffmail.com*
2. *Assistant Professor of Mathematics, Saradha Gangadharan College, Puducherry-605 004. E-mail: mathkalpana@gmail.com*

Abstract - Let $(P,\wedge)$ be a meet-semilattice and let $S = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ be a subset of P. Then S is an A-set if $A = \{x_i \wedge x_j \ / \ x_i \neq x_j\}$ is a chain. If $f(x_i \wedge x_j) = 2^{x_i \wedge x_j} - 1$ then the n × n meet matrix obtained is called the Mersenne meet matrix on S. A recursive structure theorem for Mersenne meet matrices with A-sets on exponential divisor closed set is verified and a recursive formula for det $(S)_f$ and for $(S_f)^{-1}$ on A-sets is also verified.

Keywords - Meet Matrices, Mersenne Meet Matrices, a- Set, A-Set, Exponential divisors, Exponential divisor closed set

1. INTRODUCTION

Let $(P,\leq) = (P,\vee,\wedge\_)$ be a locally finite lattice, let $S = (x_1, x_2, \ldots\ldots\ldots\ldots x_n)$ be a subset of P and let $f : P \rightarrow \mathbf{C}$ be a function. The meet matrix $(S)_f$ on S with respect to f are defined as $((S)_f)ij = f(x_i \wedge x_j)$.

Haukkanen [4] introduced meet matrices $(S)_f$ and obtained formulae for $\det(S)_f$ and $(S_f)^{-1}$ (see also [13] and [14]). Korkee and Haukkanen [9] used incidence functions in the study of meet matrices. There we obtained new upper and lower bounds for $\det(S)_f$ and a new formula for $(S_f)^{-1}$ on meet-closed sets S (i.e., xi, xj $\in$ S $\Rightarrow$ xi $\wedge$ xj $\in$ S). Korkee and Haukkanen [12] presented a new method for calculating $\det(S)_f$ and $(S_f)^{-1}$ on those sets S which are not necessarily meet-closed.

We say that S is an ***A*-set** if the set $A = \{x_i \wedge x_j / x_i \neq x_j\}$ is a chain (an A-set need not be meet-closed). For example, chains and a-sets (with $A = \{a\}$) are known trivial A sets. Since the method, presented in [12], adapted to A-sets might not be sufficiently effective, we give a new structure theorem for $(S)_f$ where S is an A-set. One of its features is that it supports recursive function calls.
By the structure theorem we obtain a recursive formula for $\det(S)_f$ and for $(S_f)^{-1}$ on A-sets. By dissolving the recursion on certain sets we also obtain the known explicit determinant and inverse formulae on chains and a-sets.

$(\mathbf{Z}+,|) = (\mathbf{Z}+, \text{gcd}, \text{lcm})$ is a locally finite lattice, where | is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. Thus meet matrices are generalizations of GCD matrices $((S)_f)_{ij} = f(\gcd(x_i, x_j))$. For general accounts of GCD matrices, see [6]. Meet matrices are also generalizations of GCUD matrices, the unitary analogies of GCD matrices, see [5]. Thus the results also hold for GCUD matrices.

2. DEFINITIONS

Let $(P, <) = (P, \wedge)$ be a meet-semilattice and let S be a nonempty subset of P. We say that S is meet-closed if $x \wedge y \in S$ whenever $x, y \in S$. We say that S is lower-closed if $(x \in S; y \leq x) \Rightarrow y \in S$ holds for every $y \in P$. It is clear that a lower-closed set is always meet-closed but the converse is not true.
The method used requires that we arrange the elements of S analogously to the elements of chain A.

Definition 2.1

T*he binary operation* $\sqcap$ *is defined by*

$$S_1 \sqcap S_2 = [x \wedge y / x \in S_1, y \in S_2, x \neq y] \quad (2.1)$$

where S_1 *and* S_2 *are nonempty subsets of* P.
Let S *be a subset of* P *and let* $a \in P$. *If* $S \sqcap S = \{a\}$, *then the set* S *is said to be an* ***a-set***.

Definition 2.2

Let $S = \{ x_1, x_2, \dots\dots\dots\dots x_n \}$ *be a subset of P with* $x_i < x_j \Rightarrow i < j$ *and let* $A = \{ a_1, a_2, \dots\dots\dots\dots a_{n-1} \}$ *be a multichain (i.e. a chain where duplicates are allowed) with* $a_1 \le a_2 \le \dots\dots\dots \le a_{n-1}$.
The set S is said to be an ***A-set*** *if*
$\{x_k\} \sqcap \{x_{k+1}, \dots\dots, x_n\} = \{a_k\}$ *for all* $k = 1, 2, \dots\dots, n-1$.
Every chain $S = \{ x_1, x_2, \dots\dots\dots\dots x_n \}$ is an A-set with $A = S \setminus \{x_n\}$ and every a-set is always an A-set with $A = \{a\}$.

Definition 2.3

An integer $d = \prod_{i=1}^{t} p_i^{a_i}$ is said to be an **exponential divisor** of $m = \prod_{i=1}^{t} p_i^{b_i}$, if ai | bi for every $1 \le i \le t$ and is denoted by $d \mid_e m$.
A set $S = \{x_1, x_2, x_3, \dots, x_n\}$ is said to be an exponential divisor closed set if the exponential divisors of every element of S belongs to S. For example {12, 18, 36} is not an exponential divisor closed set. But, {6, 12, 18, 36} is an exponential divisor closed set.

Definition 2.4

Let f be a complex-valued function on P. Then the $n \times n$ *matrix* $(S)_f$, *where* $((S)_f)_{ij} = f(x_i \wedge x_j)$, *is called the meet matrix on S with respect to f. Also the* $n \times n$ *matrix* $(S)_f$ *,where*
$((S)_f)_{ij} = f(x_i \wedge x_j) = 2^{x_i \wedge x_j} - 1$, *is called the Mersenne meet matrix.*

3. MERSENNE MEET MATRICES ON *A*-SETS

3.1 Structure Theorem

Theorem 3.1 (Structure Theorem)
Let $S = \{ x_1, x_2, \dots\dots\dots\dots x_n \}$ *be an A-set, where* $A = \{ a_1, a_2, \dots\dots\dots\dots a_{n-1} \}$ *is a multichain. Let* $f_1, f_2, \dots\dots, f_n$ *denote the functions on P defined by* $f_1 = f$ *and*

$$f_{k+1}(x) = f_k(x) - \frac{f_k(a_k)^2}{f_k(x_k)} \qquad (3.1)$$

for $k = 1, 2, \dots\dots\dots, n-1$.
Then

$$(S)_f = M^T DM, \qquad (3.2)$$

where $D = \text{diag}(f_1(x_1), f_2(x_2), \dots\dots\dots, f_n(x_n))$ *and M is the* $n \times n$ *upper triangular matrix with 1's on its main diagonal, and further*

$$(M)_{ij} = \frac{f_i(a_i)}{f_i(x_i)} \qquad (3.3)$$

for all $i < j$. *(Note that* $f_1, \dots\dots\dots, f_n$ *exist if and only if* $(f_k(x_k) = 0, a_k \ne x_k) \Rightarrow f_k(a_k) = 0$
holds for all $k = 1, 2, \dots\dots\dots, n-1$. *In the case* $f_k(a_k) = f_k(x_k) = 0$ *we can write e.g.* $(M)_{kj} = 0$
for all $k < j$.
Proof : Let $i < j$. Then

$$(M^T DM)_{ij} = \sum_{k=1}^{n} (M)_{ki}(D)_{kk}(M)_{kj}$$

$$= f_i(a_i) + \sum_{k=1}^{i-1} \frac{f_k(a_k)^2}{f_k(x_k)} \qquad (3.4)$$

$$= f_i(a_i) + \sum_{k=1}^{i-1} (f_k(a_i) - f_{k+1}(a_i))$$
$$= f_1(a_i) = f(x_i \wedge x_j).$$

The case $i = j$ is similar, we only replace every a_i with x_i in (3.4). Since $M^T DM$ is symmetric, we do not need to treat the case $i > j$.

3.2 Determinant of Meet matrix on *A*-sets

By Structure Theorem we obtain a new recursive formula for $\det(S)_f$ on A-sets.

Theorem 3.2 *Let* $S = \{ x_1, x_2, \dots\dots\dots\dots x_n \}$ *be an A-set, where* $A = \{ a_1, a_2, \dots\dots\dots\dots a_{n-1} \}$ *is a multichain. Let* $f_1, f_2, \dots\dots, f_n$ *be the functions defined in* (3.1). *Then*

$$\det(S)_f = f_1(x_1) f_2(x_2) \dots\dots\dots f_n(x_n), \qquad (3.5)$$

By Theorem 3.2 we obtain a known explicit formula for $\det(S)_f$ on chains presented in [4, Corollary 3] and [14, Corollary 1].

Corollary 3.1 *If* $S = \{x_1, x_2, \dots\dots\dots\dots x_n\}$ *is a chain, then* $\text{Det}(S)_f = f(x_1) \prod_{k=2}^{n} (f(x_k) - f(x_{k-1}))$ (3.6)
Proof : By Theorem 3.2 we have
$\det(S)_f = f_1(x_1) f_2(x_2) \dots\dots\dots f_n(x_n)$, where
$f_1 = f$ and $f_{k+1}(x) = f_k(x) - f_k(x_k) = f(x) - f(x_k)$ for all
$k = 1, 2, \dots\dots\dots\dots\dots, n-1$. This completes the proof.

By Theorem 3.2 we also obtain a known explicit formula for $\det(S)_f$ on a-sets. This formula has been presented (with different notation) in [4, Corollary of Theorem 3] and [12,Corollaries 5.1 and 5.2], and also in [2, Theorem 3] in number-theoretic setting.
The case $f(a) = 0$ is trivial, since then
$(S)_f = \text{diag}(f(x_1), f(x_2), \dots\dots, f(x_n))$
and $\det(S)_f = f(x_1) f(x_2) \dots\dots f(x_n)$.

Corollary 3.2 *Let* $S = \{ x_1, x_2, \dots\dots\dots\dots x_n \}$ *be an a-set, where* $f(a) \ne 0$.
If $a \in S$ *(i.e.* $a = x_1$*), then*

$$\det(S)_f = f(a)(f(x_2) - f(a)) \dots (f(x_n) - (a)). \qquad (3.7)$$

If $a \notin S$, then

$$\det(S)_f = \sum_{k=1}^{n} \frac{f(a)(f(x_1)-f(a)).....(f(x_n)-f(a))}{f(x_k)-f(a)} + (f(x_1)-f(a))\ldots.(f(x_n)-f(a)). \quad (3.8)$$

Example 3.1 Let $(P, \leq\cdot) = (\mathbf{Z}+,|)$ and $S = \{2,4,16\}$.

Then $S=\begin{bmatrix} 2^2-1 & 2^2-1 & 2^2-1 \\ 2^2-1 & 2^4-1 & 2^4-1 \\ 2^2-1 & 2^4-1 & 2^{16}-1 \end{bmatrix}$.

As S is an A-set with the chain $A=\{2,4\}$ by (3.1) we have $f_1 = f$, $f_2(x) = f_1(x) - f_1(2)^2/f_1(2)$ and $f_3(x) = f_2(x) - f_2(4)^2/f_2(4)$.

Let $f(x) = 2^x - 1$. Then

$f_1(x) = 2^x - 1$, $f_2(x) = 2^x - 4$, $f_3(x) = 2^x - 16$

and by Theorem 3.1 $(S)_f = M^T DM$, where

$D=\mathrm{diag}(3,12,65520)$ and $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

and by Theorem 3.2 we have

$\det(S)_f = f_1(2)f_2(4)f_3(16) = 3(12)(65520)$
$= 23,58,720$.

3.3 Inverse of Mersenne meet matrix on *A*-sets

By Structure Theorem we obtain a new recursive formula for $(S_f)^{-1}$ on A-sets.

Theorem 3.3 *Let $S = \{x_1, x_2,\ldots\ldots\ldots\ldots x_n\}$ be an A-set, where $A = \{a_1, a_2,\ldots\ldots\ldots\ldots a_{n-1}\}$ is a multichain. Let $f_1, f_{2,\ldots\ldots}, f_n$ be the functions defined in* (3.1), *where $f_i(x_i) \neq 0$ for $i = 1,2,,\ldots.,n$.*

Then $(S)_f$ is invertible and $(S_f)^{-1} = N\triangle N^T$ (3.9)

where $\triangle = \mathrm{diag}(1/f_1(x_1), 1/f_2(x_2),\ldots, 1/f_n(x_n))$ *and N is the n×n upper triangular matrix with 1's on its main diagonal, and further*

$$(N)_{ij} = -\frac{f_i(a_i)}{f_i(x_i)} \prod_{k=i+1}^{j-1}\left(1-\frac{f_k(a_k)}{f_k(x_k)}\right) \quad (3.10)$$

for all $i < j$.

Proof: By Structure Theorem $(S)_f = M^T DM$, where M is the matrix defined in (3.3) and $D = \mathrm{diag}(f_1(x_1), f_2(x_2),\ldots\ldots.., f_n(x_n))$.

Therefore $(S_f)^{-1} = N\triangle N^T$,

where $D^{-1} = \mathrm{diag}(1/f_1(x_1), 1/f_2(x_2),\ldots, 1/f_n(x_n))$ and $M^{-1} = N$ is the $n \times n$ upper triangular matrix in 3.10.

Example 3.1.1

S is considered the same as in Example 3.1 then by $(S_f)^{-1} = N\triangle N^T$,

$\triangle = \mathrm{diag}(1/3, 1/12, 1/65520)$) , $N = M^{-1}$,

$N=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$,

$$(S_f)^{-1} = \begin{bmatrix} \frac{5}{12} & \frac{-1}{12} & 0 \\ \frac{-1}{12} & \frac{5461}{65520} & \frac{-1}{65520} \\ 0 & \frac{1}{65520} & \frac{1}{65520} \end{bmatrix}$$

Corollary 3.3 *Let $S = \{x_1, x_2,\ldots\ldots\ldots\ldots x_n\}$ be an a-set, where $f(a) \neq 0$ and $f(x_k) \neq f(a)$ for all $k = 2,\ldots\ldots., n$. If $a \in S$ (i.e. $a = x_1$), then $(S)_f$ is invertible and*

$$\left((S_f)^{-1}\right)_{ij} = \begin{cases} \frac{1}{f(a)} + \sum_{k=2}^{n} \frac{1}{f(x_k)-f(a)} & if\ i=j=1, \\ \frac{1}{f(x_k)-f(a)} & if\ 1<i=j, \\ \frac{1}{f(a)-f(x_k)} & if\ 1=i<j=k\ or\ 1=j<i=k \\ 0 & otherwise \end{cases} \quad (3.11)$$

If $a \notin S$ and further $f(x_1) \neq f(a)$ and $\frac{1}{f(a)} \neq \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}$, then $(S)_f$ is invertible and

$$\left((S_f)^{-1}\right)_{ij} = \begin{cases} \frac{1}{f(x_k)-f(a)} - \frac{1}{[f(x_k)-f(a)]^2}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i=j, \\ \frac{1}{[f(x_k)-f(a)][f(x_k)-f(a)]}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i \neq j. \end{cases} \quad (3.12)$$

4. CONCLUSION

In this paper we prove by examples that the Mersenne Meet matrices with A sets on exponential divisor closed set satisfies structure theorem and calculate the determinant and inverse of the matrix through the results based on A sets.

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